SHELLS ON ELASTIC FOUNDATIONS

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CERTIFICATE

Certified that the thesis 'Shells on Elastic Foundations' is the bonafide work done by Mr. K.K. Kedia under my guidance and has not been submitted for the award of any degree elsewhere.

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Kamal Kumar Kedia

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LIST OF SYMBOLS

a	-	Outer radius of the shell
a_{1f}, b_{1f}, etc.	-	Constant coefficients characterising foundation model [Eqns.(3.14),(4.6)]
A, B	-	Coefficients of first quadratic equation for the surface of the shell
dA	-	Elemental area
b	-	Radius of the column
D	-	Flexure rigidity of the shell
$e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\gamma\gamma}$	-	Normal strains in foundation
$e_{\alpha\beta}$, $e_{\beta\gamma}$, $e_{\gamma\alpha}$	-	Shear strains in foundation
E, μ	-	Elastic constants of the foundation material
E_o, μ_o	-	Elastic constants defined by Eqns.(3.15)
E_s, μ_s	-	Elastic constant of the shell
h	-	Thickness of the shell
H	-	Thickness of the foundation
H_1, H_2, H_3	-	Coefficients of first quadratic equation defined by Eqn. (2.3)
(h/R)	-	Parameter of spherical shell accounting for thinness
k_1, k_2	-	Curvatures of the shell

K	-	Gaussian Curvature
M_1, M_2, M_{12}, M_{21}	-	Bending moments and twisting moments in shell
N_1, N_2, S_1, S_2	-	Normal and shear forces in shell
P_α, P_β, P_γ	-	Component of body forces
$\bar{P}$	-	Load parameter
q	-	Dimensionless foundation reaction
Q_1, Q_2	-	Transverse forces in shell
R		Radius of curvature of spherical shell
R/a	-	Parameter of the shell accounting for shallowness
u, v, w	-	Displacement components in α, β, and γ directions
δu, δv, δw	-	Virtual displacements in α, β and γ directions
u_j, v_j, w_k	-	Generalised displacements in α, β and γ directions
u_s, v_s, w_s		Shell displacements in α, β and γ directions
$\bar{u}_f$, $\bar{v}_f$, $\bar{w}_f$	-	Dimensionless displacement function for flank in α, β and γ directions
$\tilde{u}_1$, $\tilde{v}_1$, $\tilde{w}_1$, $\tilde{u}_s$, $\tilde{v}_s$, $\tilde{N}_1$, $\tilde{N}_2$, $\tilde{S}$, $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_{12}$, $\tilde{q}$	-	Responses per unit load parameter

X, Y, Z	.	Projections of the load in α, β and γ directions
α, β, γ	.	Curvilinear coordinates
$\sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\gamma\gamma}$	.	Normal forces in foundation
$\sigma_{\alpha\beta}, \sigma_{\beta\gamma}, \sigma_{\gamma\alpha}$		Shear forces in foundation
$d\Omega$		Elemental volume
ϕ_1, χ_1, ψ_1		Dimensionless displacement functions for normal direction
λ, G		Lame's elastic constants
Θ		Angle made by the generators and the axis of the conical shell
$\varepsilon_1, \varepsilon_2$		Normal strain in shell
ω	..	Shear strains in shell
ξ_1, ξ_2		Associated with bending deformations
τ	..	Relative twisting deformation
∇^2		Laplace operator

SYNOPSIS

For developing countries like India, where material to labour cost is high, shell foundation for columns will play a vital role in saving material in construction. Shells on elastic foundations are of recent origin and no realistic analytical work has been done as yet although, some experimental work has been reported in the literature. A review of the work done on shells on elastic foundations has been given in Chapter I.

Various foundation models have been proposed by several authors. A comprehensive review of the various foundation models has also been given in Chapter I. The earliest and simplest formulation of elastic foundation was by Winkler who assumed that the foundation model consists of closely spaced independent linear springs. This model lacks continuity. Most consistent formulation would be to consider the foundation as a semi-infinite elastic continuum, but the analysis in this case is very complex. Several authors have proposed foundation models which are a compromise between the above mentioned two extremes. These foundation models involve more than one parameter to characterise the foundation medium. To mention a few we have models proposed by Wieghardt, Filonenko-Borodich, Pasternak,

Reissner, Hetenyi, Vlasov and Leontov and Kameswar Rao, Das and Anandakrishnan.

Using variational method a foundation model in curvilinear orthogonal coordinates has been proposed in Chapter II. This foundation model includes displacements in all the three orthogonal coordinate directions, hence making it possible to use the proposed model for foundation of any type of structure and in any set of orthogonal coordinate system. This model is capable of analysing foundations of both finite and infinite thicknesses. Multi-segment technique, which has been used to solve the boundary value problems has also been described in this chapter.

In Chapter III proposed foundation model has been used to analyse thin shallow spherical and conical shells on elastic foundations under axisymmetric normal load, transferred by a column. Three important types of boundary conditions namely simply supported, fixed or built in and free are considered. In case of free boundary condition the flank portion (foundation beyond the shell) has been taken into account. Depth of foundation has been considered as finite in the numerical examples. In case of spherical shells, two parameters accounting for thinness and shallowness of the shell, are considered. In case of conical shell only one parameter which accounts for the shallowness of the

shell has been considered. Results are presented in non-dimensional form so that they can be used for design purposes. Curves for moments, displacement, inplane forces and foundation reactions are presented for different values of parameters.

In Chapter IV, proposed foundation model has been used to analyse thin shallow spherical and conical shells on elastic foundations subjected to antisymmetric loads. Loads in radial direction and moments in radial direction transferred by the column have been considered. Simply supported, fixed and free boundary conditions have been used. Depth of foundation has been taken as finite in the numerical examples that are solved. Same parameters as mentioned in Chapter III are accounted for. The results are presented in nondimensional form. Curves for moments, inplane forces, displacements and foundation reactions have been presented for different values of parameters.

Conclusions and suggestions for future research have been made in Chapter V.

CHAPTER 1

INTRODUCTION

1.1 GENERAL:

For developing countries like India, where material to labour cost is high, shell foundation for columns will plays a vital role in saving material in construction works. Shell structure for roofs have become very popular for last 25 years because of the inherent property of its being structurally more efficient than any other type of roof structure.

Shells on elastic foundations is of recent origin. Shell foundation is of considerable importance in structural engineering because of many reasons. To mention a few (i) the material saving is upto 50 percent in many cases of construction as pointed out by Ruhle [1] and Nainan [2], (ii) the structure is more stable [2,3], (iii) contact pressure distribution is better [4], and, (iv) this type of foundation is very suitable for the foundation on weak soils.

The problem of shell foundation is useful not only in Civil Engineering but also in aerospace structures, solid propellent rocket motors, pressure vessels etc.

The problem of structure-foundation interaction is generally solved by incorporating the reaction from the foundation on the structure. This is done by idealising foundation by a simple mathematical model. Even if the foundation medium is complex, in a majority of cases, the response of the structure at the contact surface is of prime interest and hence, it would be of immense help in the analysis, if the foundation can be represented by a simple mathematical model, without forgoing the desired accuracy. To accomplish this objective, many authors have proposed foundation models; a brief review of which is given below.

1.2 BRIEF REVIEW OF THE FOUNDATION MODELS:

More than hundred years ago in 1867 Winkler [5] proposed a foundation model, known by his name as Winkler's model. He assumed the foundation model to consist of closely spaced independent linear springs. If such a foundation is subjected to a partially distributed surface loading 'p' the springs will not be affected beyond the loaded region [Fig. (1.1)]. In actual cases the foundation is observed to have the surface deformations beyond the loaded region [Fig. (1.2)]. It can be seen that Winkler's model essentially suffers from a complete 'Lack of Continuity' in the supporting medium. The load-deflection equation for this case can be written as,

$$p = kw \tag{1.1}$$

where k is the spring constant and is often referred to as 'Foundation modulus', w is the vertical deflection of the contact surface and p is the load. Although this model leads to inconsistencies, it is by far the simplest model. Even after a century this model is used for solving variety of problems, though their usefulness is limited.

Other extreme is to assume the foundation medium as a continuous elastic solid. Using this hypothesis physical behaviour of an actual foundation can be closely simulated. But the amount of mathematical complexities that creep in make this method unusable for practical engineering problems. Despite several mathematical complexities, solutions were presented on this basis [6,7], which, however, were limited to simple cases.

The need for bridging the gap between these two extreme and limiting cases and to arrive at a physically close and mathematically simple foundation model has been felt for sometime. Several authors have proposed foundation models which involve more than one parameter for the characterisation of the supporting medium.

Filonenko-Borodich [8] modified the Winkler's foundation by connecting the top ends of the spring by a stretched elastic membrane subjected to a constant tension field 'T', thus providing continuity in the foundation (Fig. 1.3). The equilibrium in the vertical direction yields the equation,

$$p = kw - T\nabla^2 w \qquad (1.2)$$

where p is the distributed normal load applied on the surface of the foundation, w is normal deflection of the surface, ∇^2 is the Laplace operator, k and T are the two parameters characterising the foundation. Hetenyi [9,10] achieved the continuity in the Winkler's foundation model by embedding, an elastic beam in the two dimensional case, and an elastic plate in the three dimensional case, with the stipulation that the hypothetical beam or plate deforms in bending only [Fig.(1.3)]. In this case the relation of load p with deflection of the surface w, can be derived as,

$$p = kw + D\nabla^4 w \qquad (1.3)$$

where D is the flexural rigidity of the embedded beam or plate, k is the spring constant and ∇^4 is the biharmonic operator.

Pasternak [11] suggested a foundation model by providing for shear interaction between the Winkler's springs as shown in Fig. (1.3). The shear interaction between the springs

has been achieved by connecting the ends of the springs to a beam or a plate (depending on as to whether it is a two dimensional or three dimensional case), consisting of incompressible vertical elements, which hence deform in transverse shear only. The differential equation relating the load p with the deflection of the surface w, can be derived as,

$$p = kw - \mu\nabla^2 w \tag{1.4}$$

where μ is the shear modulus, k is the spring constant and ∇^2 is the Laplace operator. It can be seen that this foundation model also consists of two parameters k and μ, and is equivalent to the model proposed by Filonenko-Borodich[Eq.(1.2)]. Similar model has been proposed by Wieghardt [12].

Pasternak proposed another foundation model [13] consisting of two layers of springs connected by a shear layer in between [Fig. (1.4)]. The relation between the load p and the deflection of the surface w can be expressed as

$$(1 + \frac{k}{c})\, p - \frac{\mu}{c}\nabla^2 p = kw - \mu\nabla^2 w \tag{1.5}$$

where c and k are the spring constants of the upper and lower layers of springs and μ is the shear modulus of the shear layer.

In all the above models, the Winkler's model has been modified by providing for some interaction between the spring elements and hence assuming the continuity of the foundation to some degree, which is not there in Winkler's model. In contrast to these, starting from the elastic-continuum theory, and introducing simplifying assumptions with respect to the expected stresses and displacements, some models were proposed. One such contribution was from Reissner [14], who assumed that the in-plane stresses, σ_x, σ_y and τ_{xy} are negligible throughout the foundation layer. Also the horizontal displacements at the upper and lower surfaces of the foundation layer were assumed to be zero. Proceeding with above assumptions and solving the elastic continuum equations, the equation relating the applied distributed surface load p and the resulting surface displacements w, has been derived as,

$$c_1 w - c_2 \nabla^2 w = p - \frac{c_2}{4c_1} \nabla^2 p \qquad (1.6)$$

where $c_1 = E/H$ and $c_2 = H\mu/3$. E and μ are the elastic constants of the foundation material and H is the thickness of the foundation layer. It can be seen that Eqns. (1.5) and (1.6) are similar. Also, for constant and linearly varying loads, this equation can be seen to be mathematically equivalent to equations (1.2) and (1.4) hence establishing the

similarity of the models. In this case, neglecting the in-plane stresses, it can be shown that shear stresses τ_{xz} and τ_{zy} are constant throughout the depth of the foundation for a given surface point, which is inconsistant with the actual foundation performance, especially for thick foundation layers.

Vlasov and Leontev [15] have developed a foundation model starting from elastic continuum theory. Using Vlasov's general variational method, the load displacement relations can be derived as,

$$p = kw - 2t\nabla^2 w \tag{1.7}$$

where p is the distributed surface load and w is the normal deflection. k and t are the two parameters characterising the foundation and can be expressed in terms of the elastic constants of the material and geometric properties of the foundation layer. Comparing Eqn. (1.7) with Eqns. (1.2) and (1.4) it can be observed that, this model is equivalent to the models proposed by Filonenko-Borodich, Pasternak and Wieghardt.

Kameswar Rao, Das and Anandakrishnan [16] presented a foundation model starting from elastic-continuum theory and incorporating horizontal displacements of the elastic foundation. Variational approach was used to find the relation between load and foundation displacements u, v and w.

A close examination of the various models reviewed above, reveals that these models fall short for direct application to practical problems, either because the analysis is cumbersome or because the assumptions made for natural foundation media cannot be fully justified. Moreover, the models discussed above cannot suitably be applied to shell foundation problems.

1.3 BRIEF REVIEW OF SHELLS ON ELASTIC FOUNDATIONS:

Some work has been reported in the literature about the construction of shell structures for foundation purposes but no work has been reported on the analysis of shells on elastic foundations. Shells for foundations, so far, have been designed for an assumed soil reaction. No analysis has been done to find out the actual distribution of soil reaction under the shell when the shell is subjected to axisymmetric and antisymmetric types of loads.

Felix Candela [17] is the first to report about shells being used as foundation structure in Mexico. He used hyperbolic paraboloid shells. The main reason for him to use shell was to properly distribute the load on a weak soil.

Shells for foundation purposes have also been used in India and in some other countries. Banerjee [18] and Kaimal [19]

have reported works done in India using shell as foundation structure. To mention a few, hypar footing was constructed in Calcutta for building works because the site had soil of filled up nature. A water tower of 15,000 gallons capacity was built at Adityapur, Bihar, on a single column of 51' height. The base was subjected to direct load, wind load and earthquake moments. A hypar footing of 30 cms thickness covering an area of 6 m x 6 m was adopted. A dome structure was built for foundation for an intake well at Hatia dam. Hypar footings were provided in Madras for a housing project.

Sondhi and Patel [20] have designed hypar footings for a building works in Mombase, Kenya. Conical shells have been used for foundations of towers and tall chimneys at Stuttgart, Dresden, West Berlin, Hamburg, Hannover and Kulpenberg as reported by Ruhle [1] and Leonhardt [21]. The need for shell foundations for tower shaped structures have been stressed in their papers. Using membrane analysis and considering soil reaction to be vertical and uniformly distributed, Kurian [22, 23] has calculated the ultimate strength of hypar footings subjected to axisymmetric loads. Principle of virtual work to a general mechanism of failure was used.

Concrete spherical shells subjected to axisymmetric and anti-symmetric loadings have been analysed; using membrane theory, by Sharma [24] and Sharma and Mawal [3]. The

soil reaction has been taken as uniformly distributed in axisymmetric case and a function of Cos θ in antisymmetric case. Using variational method, Vlasov and Leontov [15] have solved axisymmetric problem of thin shallow spherical shell on elastic foundation. They considered the normal displacement in foundation medium.

Model studies have been made by many authors on shells on elastic foundations [25,26,27,4]. Nicholls and Izadi [25] have tested cone models and hypar models on sand foundations. Measured strains and soil reactions are reported. The results obtained experimentally were compared with the theoretical results obtained using membrane analysis for shells and assuming a uniform sand reaction. The material used for models were plexiglass and epoxy.

1.4 OBJECT AND SCOPE OF THE PRESENT INVESTIGATION:

An attempt is made here to improve upon the foundation models given by Vlasov and Leontev [15] and Kameswar Rao, Das and Anandakrishnan [16] by formulating the foundation model in curvilinear coordinates and considering displacements in all the three coordinate directions. The mathematical model is derived using generalised variational method to the elastic continuum. The model presented is very useful and simple for the analysis of any type of structure on elastic foundation.

In Chapter II, the equations for the proposed foundation model has been derived in orthogonal curvilinear coordinates. Vlasov's [28] shell equations are used in formulating the problem. Shell foundation interaction equations are presented in orthogonal curvilinear coordinates. Kelnins's multi segment method [29] which is used to find out the solution of the boundary value problem, is stated in brief.

In Chapter III, the generalised equations for shell foundation interactions are reduced for axisymmetric thin shallow spherical shells and conical shells on elastic foundations. These are then solved for normal load on shell transferred by the column. Results are presented for static case and for three types of boundary conditions namely, simply supported, fixed and free.

In Chapter IV, the generalised equations of Chapter II, are specialised for anti-symmetric thin shallow, spherical and conical shells. These are solved for radial load and moments transferred by a column on shell. Results are presented for static case and for three types of boundary conditions namely simply supported, fixed and free.

In Chapter V, conclusions and general recommendations for future studies have been made.

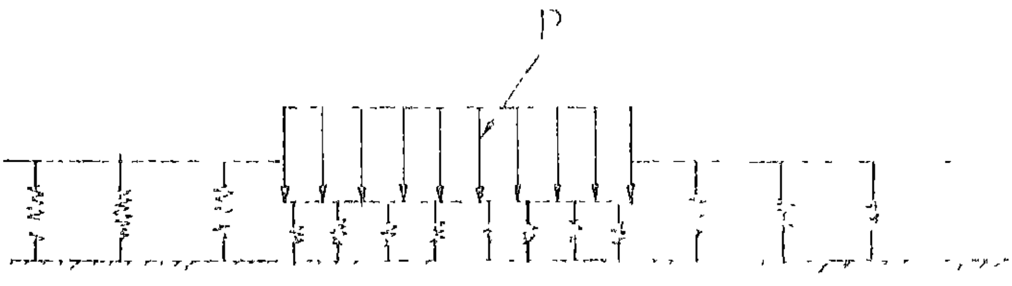


FIG. 1.1 WINKLER'S MODEL

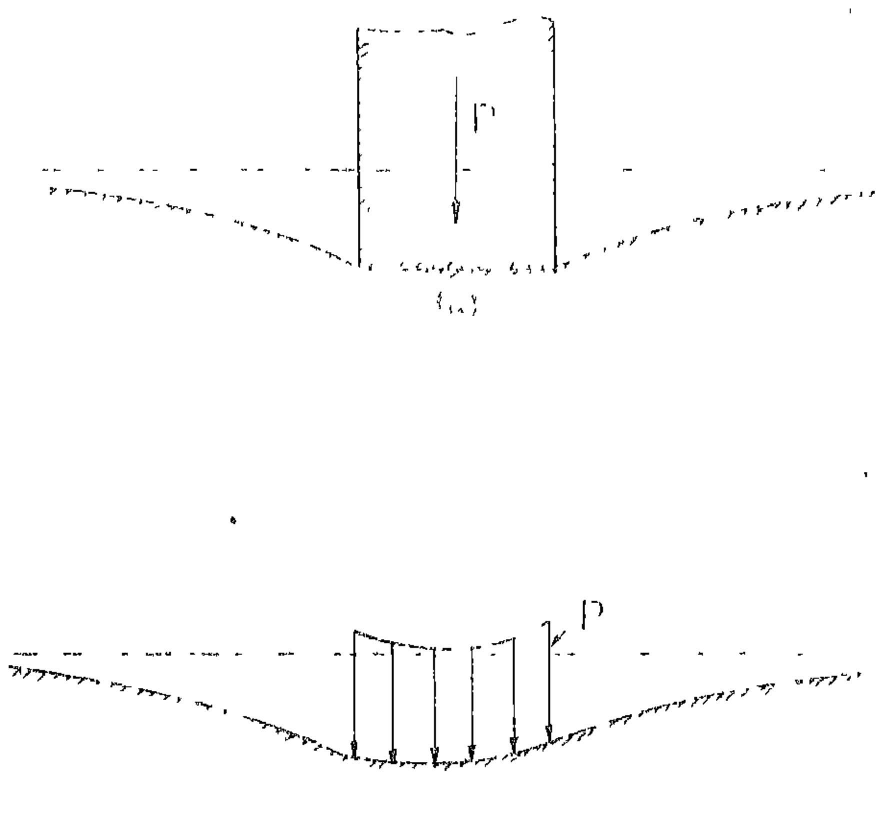
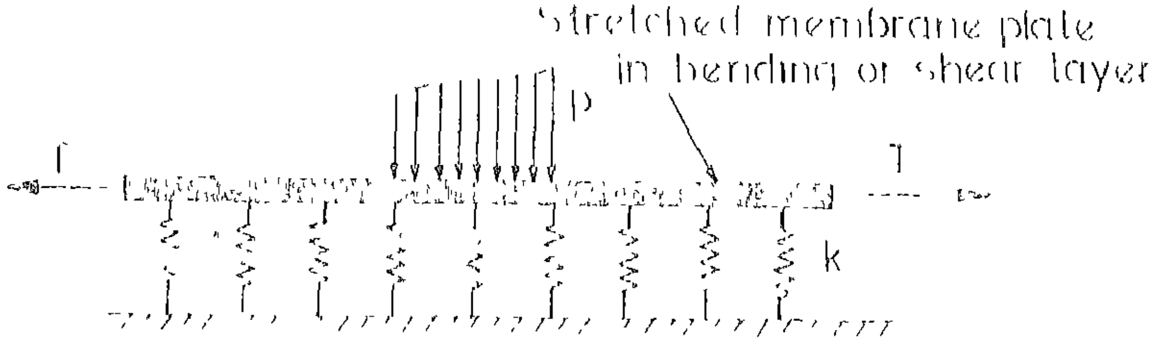


FIG. 1.2 SURFACE DEFORMATIONS BEYOND THE LOADED REGION



IG. 1.3 CONVENTIONAL SKETCH SHOWING VARIOUS FOUNDATION MODELS

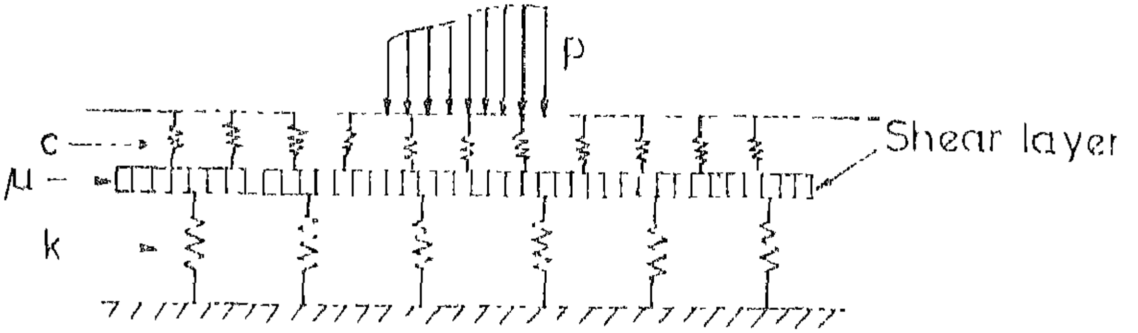


FIG. 1.4 PASTERNAK'S MODIFIED FOUNDATION MODEL

CHAPTER II

FORMULATION OF THE PROBLEM USING VARIATIONAL METHOD

2.1 GENERAL:

In this chapter, three dimensional equilibrium equations of elastic body in curvilinear coordinates are converted to two dimensional ones by the application of the principle of the virtual work. In this process, the elastic body becomes discrete in one direction but remains continuum in other two directions. This is achieved by taking the displacements as partly known in one direction and partly unknown in other two directions. The resulting two dimensional equations of the elastic body can be specialized for an elastic foundation on which an elastic shell can rest. By the help of an appropriate free body diagram for the shell the equilibrium equations for the shell can be modified to include the elastic foundation effect. For this purpose Vlasov's shallow shell equations are used for the shell portion.

To solve the equations of the shell on elastic foundation, shooting method, commonly known as multisegment method to Structural Engineers has been used and this method is described in brief at the end of this chapter.

2.2 FOUNDATION MODEL:

2.2.1 Basic Equations for Elastic Foundation:

Consider a small element of an elastic body in orthogonal coordinate system α, β and γ as shown in Fig. (2.1). $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$ and $\sigma_{\gamma\gamma}$ are the normal stresses and $\sigma_{\alpha\beta}$, $\sigma_{\beta\gamma}$, and $\sigma_{\gamma\alpha}$ etc. are the shear stresses. Let P_α, P_β and P_γ be the components of the body forces per unit volume in α, β and γ directions respectively. By the principle of virtual work,

$$\int_\Omega \Big[\frac{\partial}{\partial\alpha}(\sigma_{\alpha\alpha} H_2H_3) + \frac{\partial}{\partial\beta}(\sigma_{\alpha\beta} H_1H_3) + \frac{\partial}{\partial\gamma}(\sigma_{\alpha\gamma} H_1H_2) - \sigma_{\beta\beta} H_3 \frac{\partial H_2}{\partial\alpha} - \sigma_{\gamma\gamma} H_2 \frac{\partial H_3}{\partial\alpha} + \sigma_{\beta\alpha} . H_3 \frac{\partial H_1}{\partial\beta} + \sigma_{\gamma\alpha} H_2 \frac{\partial H_1}{\partial\gamma} + P_\alpha H_1H_2H_3 \Big] \delta u \, d\Omega = 0 \qquad (2.1a)$$

$$\int_\Omega \Big[\frac{\partial}{\partial\beta}(\sigma_{\beta\beta} H_3H_1) + \frac{\partial}{\partial\gamma}(\sigma_{\beta\gamma} H_1H_2) + \frac{\partial}{\partial\alpha}(\sigma_{\beta\alpha} H_2H_3) - \sigma_{\gamma\gamma} H_1 \frac{\partial H_3}{\partial\beta} - \sigma_{\alpha\alpha} H_3 \frac{\partial H_1}{\partial\beta} + \sigma_{\gamma\beta} H_1 \frac{\partial H_2}{\partial\gamma} + \sigma_{\alpha\beta} H_3 \frac{\partial H_2}{\partial\alpha} + P_\beta H_1H_2H_3 \Big] \delta v \, d\Omega = 0 \qquad (2.1b)$$

and,

$$\int_\Omega \Big[\frac{\partial}{\partial\gamma}(\sigma_{\gamma\gamma} H_1H_2) + \frac{\partial}{\partial\gamma}(\sigma_{\gamma\alpha} H_2H_3) + \frac{\partial}{\partial\beta}(\alpha_{\gamma\beta} H_3H_1) - \sigma_{\alpha\alpha} H_2 \frac{\partial H_1}{\partial\gamma} - \sigma_{\beta\beta} H_1 \frac{\partial H_2}{\partial\gamma} + \sigma_{\alpha\gamma} H_2 \frac{\partial H_3}{\partial\alpha} + \sigma_{\beta\gamma} H_1 \frac{\partial H_3}{\partial\beta} + P_\gamma H_1H_2H_3 \Big] \delta w \, d\Omega = 0 \qquad (2.1c)$$

where $d\Omega$ is the elemental volume, H_1, H_2, H_3 are the coefficients of the first quadratic,

$$ds^2 = H_1^2\, d\alpha^2 + H_2^2\, d\beta^2 + H_3^2\, d\gamma^2 \tag{2.2}$$

where ds is the differential distance and,

$$H_1^2 = \frac{\partial \vec{r}}{\partial \alpha}\frac{\partial \vec{r}}{\partial \alpha}\ ;\ H_2^2 = \frac{\partial \vec{r}}{\partial \beta}\frac{\partial \vec{r}}{\partial \beta}\ ;\ H_3^2 = \frac{\partial \vec{r}}{\partial \gamma}\frac{\partial \vec{r}}{\partial \gamma} \tag{2.3}$$

where r is the radius vector.

The displacement functions can be presented in a finite series in the form,

$$u(\alpha,\beta,\gamma) = \sum_{i=1}^{l} u_i\ (\alpha,\ \beta)\ \phi_i(\gamma) \tag{2.4a}$$

$$v(\alpha,\beta,\gamma) = \sum_{j=1}^{m} v_j\ (\alpha,\ \beta)\ \chi_j(\gamma) \tag{2.4b}$$

$$w(\alpha,\beta,\gamma) = \sum_{k=1}^{n} w_k\ (\alpha,\ \beta)\ \psi_k(\gamma) \tag{2.4c}$$

where $\phi_i(\gamma)$, $\chi_j(\gamma)$ and $\psi_k(\gamma)$ are known (assumed) dimensionless displacement functions consistent with the constraints of the problem. The unknown functions of displacements $u_i(\alpha,\beta)$, $v_j(\alpha,\beta)$ and $w_k\ (\alpha,\ \beta)$ are referred as the generalised displacements and have the unit of length.

On substitution for displacement functions from Eqn. (2.4a), Eqn. (2.1a) can be written as,

$$\int_A \Big[\int_\gamma \Big\{ \frac{\partial}{\partial \alpha} (\sigma_{\alpha\alpha} \, H_2 H_3) \, \phi_f + \frac{\partial}{\partial \beta} (\sigma_{\alpha\beta} \, H_1 H_3) \, \phi_f$$

$$- \sigma_{\alpha\gamma} \, H_1 H_2 \, (H_3 \, \phi_f)' - \sigma_{\beta\beta} \, H_3 \frac{\partial H_2}{\partial \alpha} \cdot \phi_f$$

$$- \sigma_{\gamma\gamma} \, H_2 \frac{\partial H_3}{\partial \alpha} \phi_f + \sigma_{\beta\alpha} \, H_3 \frac{\partial H_1}{\partial \beta} \phi_f + \sigma_{\gamma\alpha} \, H_2 \frac{\partial H_1}{\partial \gamma} \phi_f$$

$$+ P_\alpha \, H_1 H_2 H_3 \, \phi_f \Big\} \, H_3 \, d\gamma + \Big\{ \sigma_{\alpha\gamma} \, H_1 H_2 H_3 \, \phi_f \Big\}_{\gamma_i} \Big] \delta u_f \, dA = 0$$

(2.5)

where use of the following identities have been made,

$$\int_\Omega \frac{\partial}{\partial \gamma} (\sigma_{\alpha\gamma} \, H_1 H_2) \, \delta u_f \, \phi_f \, d\Omega = \int_A [\sigma_{\alpha\gamma} \, H_1 H_2 H_3]_{\gamma_i} \, \delta u_f \, dA$$

$$- \int_\Omega \sigma_{\alpha\gamma} \, H_1 H_2 (H_3 \phi_f)' \, \delta u_f \, d\Omega$$

(2.6a)

and,

$$d\Omega = H_3 \, d\gamma \, . \, dA \qquad (2.6b)$$

where prime denotes derivative with respect to γ.

Since $\delta u_f \neq 0$, Eqn. (2.5) can be written as,

$$\int_\gamma \Big[\frac{\partial}{\partial \alpha} (\sigma_{\alpha\alpha} \, H_2 H_3) \, \phi_f + \frac{\partial}{\partial \beta} (\sigma_{\alpha\beta} \, H_1 H_3) \, \phi_f$$

$$- \sigma_{\alpha\gamma} \, H_1 H_2 \, (H_3 \phi_f)' - \sigma_{\beta\beta} \, H_3 . \frac{\partial H_2}{\partial \alpha} \phi_f - \sigma_{\gamma\gamma} \, H_2 . \frac{\partial H_3}{\partial \alpha} \phi_f$$

$$+ \sigma_{\beta\alpha} \, H_3 \frac{\partial H_1}{\partial \beta} \phi_f + \sigma_{\gamma\gamma} \, H_3 \frac{\partial H_1}{\partial \gamma} \phi_f + P_\alpha \, H_1 H_2 H_3 \, \phi_f \Big] \, H_3 \, d\gamma$$

$$+ \Big\{ \sigma_{\alpha\gamma} \, H_1 H_2 H_3 \, \phi_f \Big\}_{\gamma_i} = 0 \qquad (2.7a)$$

$$(f = 1, 2, \ldots)$$

Similarly Eqns. (2.1b) and (2.1c) can be written as,

$$\int_\gamma \Big[\frac{\partial}{\partial\beta} (\sigma_{\beta\beta} H_1H_3)\chi_g - \sigma_{\beta\gamma} H_1H_2 (H_3\chi_g)' + \frac{\partial}{\partial\alpha} (\sigma_{\alpha\beta} H_3H_2)\chi_g - \sigma_{\gamma\gamma} H_1 \frac{\partial H_3}{\partial\beta}\chi_g - \sigma_{\alpha\alpha} H_3 \frac{\partial H_1}{\partial\beta}\chi_g + \sigma_{\gamma\beta} H_1 \frac{\partial H_2}{\partial\gamma}\chi_g + \sigma_{\alpha\beta} H_3 \frac{\partial H_2}{\partial\alpha}\chi_g + P_\beta H_1H_2H_3\chi_g\Big] H_3 \, d\gamma + [\sigma_{\beta\gamma} H_1H_2H_3\chi_g]_\gamma = 0 \tag{2.7b}$$

$$(g = 1,2,\ldots m)$$

and

$$\int_\gamma \Big[- \sigma_{\gamma\gamma} H_1H_2(H_3 \psi_h)' + \frac{\partial}{\partial\alpha} (\sigma_{\gamma\alpha} H_3H_2)\psi_h + \frac{\partial}{\partial\beta} (\sigma_{\alpha\beta} H_3H_1)\psi_h - \sigma_{\alpha\alpha} H_2 \frac{H_1}{\gamma}\psi_h - \sigma_{\beta\beta} H_1 \frac{\partial H_2}{\partial\gamma}\psi_h + \sigma_{\alpha\gamma} H_2 \frac{\partial H_3}{\partial\alpha}\psi_h + \sigma_{\beta\gamma} H_1 \frac{\partial H_3}{\partial\beta}\psi_h + P_\gamma H_1H_2H_3\psi_h\Big] H_3 \, d\gamma + [\sigma_{\gamma\gamma} H_1H_2H_3\psi_h]_\gamma = 0 \tag{2.7c}$$

$$(h = 1,2,\ldots \quad)$$

The stress-strain law for isotropic Hooken material can be written as,

$$\sigma_{\alpha\alpha} = \lambda\Delta + 2G\, e_{\alpha\alpha} \tag{2.8a}$$

$$\sigma_{\beta\beta} = \lambda\Delta + 2G\, e_{\beta\beta} \tag{2.8b}$$

$$\sigma_{\gamma\gamma} = \lambda\Delta + 2G\, e_{\gamma\gamma} \tag{2.8c}$$

$$\sigma_{\alpha\beta} = G\, e_{\alpha\beta} \tag{2.8d}$$

$$\sigma_{\beta\gamma} = G\, e_{\beta\gamma} \tag{2.8e}$$

$$\sigma_{\gamma\alpha} = G\, e_{\gamma\alpha} \tag{2.8f}$$

where $\Delta =$
$$= e_{\alpha\alpha} + e_{\beta\beta} + e_{\gamma\gamma} \tag{2.9}$$

$e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\gamma\gamma}$ are the normal strains and $e_{\alpha\beta}$, $e_{\beta\gamma}$, $e_{\gamma\alpha}$ are the shear strains; λ and G are Lame's constants given as,

$$\lambda = E\mu/(1+\mu)(1-2\mu)) \tag{2.10a}$$

and
$$G = E/2(1+\mu) \tag{2.10b}$$

where E is the Young's modulus and μ is the Poisson's ratio of the material.

The strain-displacement relation can be written as,

$$e_{\alpha\alpha} = \frac{1}{H_1}\frac{\partial u}{\partial\alpha} + \frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta} v + \frac{1}{H_1H_3}\frac{\partial H_1}{\partial\gamma} w \tag{2.11a}$$

$$e_{\beta\beta} = \frac{1}{H_2}\frac{\partial v}{\partial\beta} + \frac{1}{H_2H_3}\frac{\partial H_2}{\partial\gamma} w + \frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha} u \tag{2.11b}$$

$$e_{\gamma\gamma} = \frac{1}{H_3}\frac{\partial w}{\partial\alpha} + \frac{1}{H_1H_3}\frac{\partial H_3}{\partial\alpha} u + \frac{1}{H_3H_2}\frac{\partial H_3}{\partial\beta} v \tag{2.11c}$$

$$e_{\alpha\beta} = \frac{H_1}{H_2}\frac{\partial}{\partial\beta}\left(\frac{1}{H_1} u\right) + \frac{H_2}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{H_2} v\right) \tag{2.11d}$$

$$e_{\beta\gamma} = \frac{H_2}{H_3}\frac{\partial}{\partial\gamma}\left(\frac{1}{H_2} v\right) + \frac{H_3}{H_2}\frac{\partial}{\partial\beta}\left(\frac{1}{H_3} w\right) \tag{2.11e}$$

$$\sigma_{\gamma\alpha} = \frac{H_3}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{H_3}w\right) + \frac{H_1}{H_3}\frac{\partial}{\partial\gamma}\left(\frac{1}{H_1}u\right) \tag{2.11f}$$

Stresses in terms of generalised displacement functions u_i, v_j and w_k can be written as

$$\begin{aligned}\sigma_{\alpha\alpha} = {} & \frac{\lambda}{H_1H_2H_3}\Big[\sum_{i=1}^{l}\frac{\partial}{\partial\alpha}(H_2H_3u_i)\,\phi_i + \sum_{j=1}^{m}\frac{\partial}{\partial\beta}(H_3H_1v_j)\chi_j \\ & + \sum_{k=1}^{n}\frac{\partial}{\partial\gamma}(H_1H_2\Psi_k)\,w_k\Big] + 2G\Big[\sum_{i=1}^{l}\frac{1}{H_1}\frac{\partial u_i}{\partial\alpha}\phi_i \\ & + \sum_{j=1}^{m}\frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta}v_j\chi_j + \sum_{k=1}^{n}\frac{1}{H_1H_3}\frac{\partial H_1}{\partial\gamma}w_k\Psi_k\Big]\end{aligned} \tag{2.12a}$$

$$\begin{aligned}\sigma_{\beta\beta} = {} & \frac{\lambda}{H_1H_2H_3}\Big[\sum_{i=1}^{l}\frac{\partial}{\partial\alpha}(H_2H_3u_i)\,\phi_i + \sum_{j=1}^{m}\frac{\partial}{\partial\beta}(H_3H_1v_j)\chi_j \\ & + \sum_{k=1}^{n}\frac{\partial}{\partial\gamma}(H_1H_2\Psi_k)\,w_k\Big] + 2G\Big[\sum_{j=1}^{m}\frac{1}{H_2}\frac{\partial v_j}{\partial\beta}\chi_j \\ & + \sum_{i=1}^{l}\frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha}u_i\,\phi_i + \sum_{k=1}^{n}\frac{1}{H_2H_3}\frac{\partial H_2}{\partial\gamma}w_k\Psi_k\Big]\end{aligned} \tag{2.12b}$$

$$\begin{aligned}\alpha_{\gamma\gamma} = {} & \frac{\lambda}{H_1H_2H_3}\Big[\sum_{i=1}^{l}\frac{\partial}{\partial\alpha}(H_2H_3u_i)\,\phi_i + \sum_{j=1}^{m}\frac{\partial}{\partial\beta}(H_3H_1\,v_j)\chi_j \\ & + \sum_{k=1}^{n}\frac{\partial}{\gamma}(H_1H_2\Psi_k)\,w_k\Big] + 2G\Big[\sum_{k=1}^{m}\frac{1}{H_3}\frac{\partial\Psi_k}{\partial\gamma}w_k \\ & + \sum_{i=1}^{l}\frac{1}{H_1H_3}\frac{\partial H_3}{\partial\alpha}u_i\,\phi_i + \sum_{i=1}^{m}\frac{1}{H_2H_3}\frac{\partial H_3}{\partial\beta}v_j\chi_j\Big]\end{aligned} \tag{2.12c}$$

$$\sigma_{\alpha\beta} = G\left[\sum_{i=1}^{l} \frac{H_1}{H_2} \frac{\partial}{\partial \beta} \left(\frac{1}{H_1} u_i\right) \phi_i + \sum_{j=1}^{m} \frac{H_2}{H_1} \frac{\partial}{\partial \alpha} \left(\frac{1}{H_2} v_j\right) \chi_j \right] \tag{2.12d}$$

$$\sigma_{\beta\gamma} = G\left[\sum_{j=1}^{m} \frac{H_2}{H_3} \frac{\partial}{\partial \gamma} \left(\frac{1}{H_2} \chi_j\right) v_j + \sum_{k=1}^{n} \frac{H_3}{H_2} \frac{\partial}{\partial \beta} \left(\frac{1}{H_3} w_k\right) \psi_k \right] \tag{2.12e}$$

$$\sigma_{\gamma\alpha} = G\left[\sum_{k=1}^{n} \frac{H_3}{H_1} \frac{\partial}{\partial \alpha} \left(\frac{1}{H_3} w_k\right) \psi_k + \sum_{i=1}^{l} \frac{H_1}{H_3} \frac{\partial}{\partial \gamma} \left(\frac{1}{H_1} \phi_i\right) u_i \right] \tag{2.12f}$$

On substitution of stresses from Eqns. (2.12), Eqn (2.5) can be written as,

$$\begin{aligned}
&\int_\gamma \frac{\partial}{\partial \alpha} \Bigg[\frac{\lambda}{H_1} \Bigg\{ \sum_{i=1}^{l} \frac{\partial}{\partial \alpha} (H_2 H_3 u_i)\, \phi_i + \sum_{j=1}^{m} \frac{\partial}{\partial \beta} (H_1 H_3 v_j)\, \chi_j \\
&+ \sum_{k=1}^{n} \frac{\partial}{\partial \gamma} (H_1 H_2 w_k)\, \psi_k \Bigg\} + 2G\, H_2 H_3 \Bigg\{ \frac{1}{H_1} \sum_{i=1}^{l} \frac{\partial u_i}{\partial \alpha} \phi_i \\
&+ \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \beta} \sum_{j=1}^{m} v_j \chi_j + \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial \gamma} \sum_{k=1}^{n} w_k \psi_k \Bigg\} \Bigg] \phi_f\, H_3\, d\gamma \\
&+ \int_\gamma \frac{\partial}{\partial \beta} \Bigg[G\, H_1 H_3 \Bigg\{ \frac{H_1}{H_2} \sum_{i=1}^{l} \frac{\partial}{\partial \beta} \left(\frac{1}{H_1} u_i\right) \phi_i \\
&+ \frac{H_2}{H_1} \sum_{j=1}^{m} \frac{\partial}{\partial \alpha} \left(\frac{1}{H_2} v_j\right) \chi_j \Bigg\} \Bigg] \phi_f\, H_3\, d\gamma \\
&- \int_\gamma G\, H_1 H_2 \Bigg[\frac{H_3}{H_1} \sum_{k=1}^{n} \frac{\partial}{\partial \alpha} \left(\frac{1}{H_3} w_k\right) \psi_k + \\
&+ \frac{H_1}{H_3} \sum_{i=1}^{l} \frac{\partial}{\partial \gamma} \left(\frac{1}{H_1} \phi_i\right) u_i \Bigg] (H_3 \phi_f)'\, H_3\, d\gamma
\end{aligned}$$

$$- \int_\gamma \left[\frac{\lambda}{H_1H_2H_3} \left\{ \sum_{i=1}^{l} \frac{\partial}{\partial\alpha} (H_2H_3 u_i) \phi_i + \sum_{j=1}^{m} \frac{\partial}{\partial\beta} (H_3H_1 v_j)\chi_j \right.\right.$$

$$\left. + \sum_{k=1}^{n} \frac{\partial}{\partial\gamma} (H_1H_2\psi_k) w_k \right\} + 2G \left\{ \frac{1}{H_2} \sum_{j=1}^{m} \frac{\partial v_j}{\partial\beta} \chi_j \right.$$

$$\left.\left. + \frac{1}{H_2H_3} \frac{\partial H_2}{\partial\gamma} \sum_{k=1}^{n} w_k \psi_k + \frac{1}{H_1H_2} \frac{\partial H_2}{\partial\alpha} \sum_{i=1}^{l} u_i \phi_i \right\} \right]$$

$$\times H_3 \frac{\partial H_2}{\partial\alpha} \phi_f H_3 d\gamma - \int_\gamma \left[\frac{\lambda}{H_1H_2H_3} \left\{ \sum_{i=1}^{l} \frac{\partial}{\partial\alpha} (H_2H_3 u_i) \phi_i \right.\right.$$

$$\left. + \sum_{j=1}^{m} \frac{\partial}{\partial\beta} (H_3H_1 v_j) \chi_j + \sum_{k=1}^{n} \frac{\partial}{\partial\gamma} (H_1H_2\psi_k) w_k \right\}$$

$$+ 2G \left\{ \frac{1}{H_3} \sum_{k=1}^{n} \frac{\partial\psi_k}{\partial\gamma} w_k + \frac{1}{H_3H_1} \frac{\partial H_3}{\partial\alpha} \sum_{i=1}^{l} u_i \phi_i \right.$$

$$\left.\left. + \frac{1}{H_3H_2} \frac{\partial H_3}{\partial\beta} \sum_{j=1}^{m} v_j \chi_j \right\} \right] H_2 \frac{\partial H_3}{\partial\alpha} \phi_f H_3 d\gamma$$

$$+ \int_\gamma G \left[\frac{H_1}{H_2} \sum_{i=1}^{l} \frac{\partial}{\partial\beta} \left(\frac{1}{H_1} u_i\right) \phi_i + \frac{H_2}{H_1} \sum_{j=1}^{m} \frac{\partial}{\partial\alpha} \left(\frac{1}{H_2} v_j\right) \chi_j \right]$$

$$\times H_3 \frac{\partial H_1}{\partial\beta} \phi_f H_3 d\gamma + \int_\gamma G \left[\frac{H_3}{H_1} \sum_{k=1}^{n} \frac{\partial}{\partial\alpha} \left(\frac{1}{H_3} w_k\right) \psi_k \right.$$

$$\left. + \frac{H_1}{H_3} \sum_{i=1}^{l} \frac{\partial}{\partial\gamma} \left(\frac{1}{H_1} \phi_i\right) u_i \right] H_2 \frac{\partial H_1}{\partial\gamma} \phi_f H_3 d\gamma$$

$$+ \int_\gamma [P_\alpha H_1H_2H_3 \phi_f(\gamma)] H_3 d\gamma + [\sigma_{\alpha\gamma} H_1H_2H_3 \phi_f]_\gamma = 0$$

$$(f = 1, 2, \ldots) \qquad (2.13a)$$

$$\int_\gamma \frac{\partial}{\partial\beta}\Big[\frac{\lambda}{H_2}\Big\{\sum_{i=1}^{l}\frac{\partial}{\partial\alpha}(H_2H_3u_i)\,\phi_i + \sum_{j=1}^{m}\frac{\partial}{\partial\beta}(H_3H_1v_j)\,\chi_j$$

$$+ \sum_{k=1}^{n}\frac{\partial}{\partial\gamma}(H_1H_2\psi_k)\,w_k\Big\} + 2G\,H_1H_3\Big\{\frac{1}{H_2}\sum_{j=1}^{m}\frac{\partial v_j}{\partial\beta}\chi_j$$

$$+ \frac{1}{H_2H_3}\frac{\partial H_2}{\partial\gamma}\sum_{k=1}^{n} w_k\psi_k + \frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha}\sum_{i=1}^{l} u_i\,\phi_i\Big\}\Big]\chi_g\;H_3\,d\gamma$$

$$-\int_\gamma G\,H_2H_1\Big[\frac{H_2}{H_3}\sum_{j=1}^{m}\frac{\partial}{\partial\gamma}\Big(\frac{1}{H_2}\chi_j\Big)\,v_j + \frac{H_3}{H_2}\sum_{k=1}^{n}\frac{\partial}{\partial\beta}\Big(\frac{1}{H_3}w_k\Big)\psi_k\Big]$$

$$\times\,\chi_g'\;H_3\,d\gamma + \int_\gamma \frac{\partial}{\partial\alpha}\Big[H_3H_2G\Big\{\frac{H_1}{H_2}\sum_{i=1}^{l}\frac{\partial}{\partial\beta}\Big(\frac{1}{H_1}u_i\Big)\,\phi_i$$

$$+ \frac{H_2}{H_1}\sum_{j=1}^{m}\frac{\partial}{\partial\alpha}\Big(\frac{v_j}{H_2}\Big)\chi_j\Big\}\Big]\chi_g\;H_3\,d\gamma - \int_\gamma\Big[\frac{\lambda}{H_1H_2H_3}$$

$$\times\Big\{\sum_{i=1}^{l}\frac{\partial}{\partial\alpha}(H_2H_3\,u_i)\,\phi_i + \sum_{j=1}^{m}\frac{\partial}{\partial\beta}(H_3H_1v_j)\chi_j$$

$$+ \sum_{k=1}^{n}\frac{\partial}{\partial\gamma}(H_1H_2\psi_k)w_k\Big\} + 2G\Big\{\frac{1}{H_1}\sum_{i=1}^{l}\frac{\partial u_i}{\partial\alpha}\,\phi_i$$

$$+ \frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta}\sum_{j=1}^{m} v_j\chi_j + \frac{1}{H_1H_3}\frac{\partial H_1}{\partial\gamma}\sum_{k=1}^{n} w_k\psi_k\Big\}\Big]$$

$$\times\,H_3\frac{\partial H_1}{\partial\beta}\chi_g\;H_3\,d\gamma + \int_\gamma G\Big[\frac{H_2}{H_3}\sum_{j=1}^{m}\frac{\partial}{\partial\gamma}\Big(\frac{1}{H_2}\chi_j\Big)\,v_j$$

$$+ \frac{H_3}{H_2}\sum_{k=1}^{n}\frac{\partial}{\partial\beta}\Big(\frac{1}{H_3}w_k\Big)\psi_k\Big]\,H_1\frac{\partial H_2}{\partial\gamma}\chi_g\;H_3\,d\gamma$$

$$+\int_\gamma G\Big[\frac{H_1}{H_2}\sum_{i=1}^{l}\frac{\partial}{\partial\beta}\Big(\frac{1}{H_1}u_i\Big)\,\phi_i + \frac{H_2}{H_1}\sum_{j=1}^{m}\frac{\partial}{\partial\alpha}\Big(\frac{1}{H_2}v_j\Big)\chi_j\Big]$$

$$\times H_3 \frac{\partial H_2}{\delta \alpha} \chi_g \; H_3 d\gamma - \int_\gamma \Big[\frac{\lambda}{H_1 H_2 H_3} \Big\{ \sum_{i=1}^{l} \frac{\partial}{\delta \alpha} (H_2 H_3 \; u_i) \; \phi_i$$

$$+ \sum_{j=1}^{m} \frac{\partial}{\delta \beta} (H_1 H_3 \; v_j) \chi_j + \sum_{k=1}^{n} \frac{\partial}{\delta \gamma} (H_1 H_2 \psi_k) \; w_k \Big\}$$

$$+ 2G \Big\{ \frac{1}{H_3} \sum_{k=1}^{n} \frac{\partial \psi_k}{\delta \gamma} w_k + \frac{1}{H_1 H_3} \frac{\partial H_3}{\delta \alpha} \sum_{i=1}^{l} u_i \phi_i$$

$$+ \frac{1}{H_2 H_3} \frac{\partial H_3}{\delta \beta} \sum_{j=1}^{m} v_j \chi_j \Big\} \Big] H_1 \frac{\partial H_3}{\partial \beta} \chi_g \; H_3 d\gamma$$

$$+ \int \Gamma_\beta \; H_1 H_2 H_3 \chi_g \; H_3 d\gamma + [\sigma_{\beta\gamma} \; H_1 H_2 H_3 \chi_g]_\gamma = 0$$

$$(g = 1, 2, \ldots \quad) \qquad (2.13b)$$

and,

$$-\int_\gamma \Big[\frac{\lambda}{H_3} \Big\{ \sum_{i=1}^{l} \frac{\partial}{\delta \alpha} (H_2 H_3 \; u_i) \; \phi_i + \sum_{j=1}^{m} \frac{\partial}{\delta \beta} (H_1 H_3 v_j) \; \chi_j$$

$$+ \sum_{k=1}^{n} \frac{\partial}{\delta \gamma} (H_1 H_2 \psi_k) \; w_k \Big\} + 2G \; H_1 H_2 \Big\{ \frac{1}{H_3} \sum_{k=1}^{n} \frac{\partial \psi_k}{\delta \gamma} w_k$$

$$+ \frac{1}{H_3 H_1} \frac{\partial H_3}{\partial \alpha} \sum_{i=1}^{l} u_i \; \phi_i + \frac{1}{H_3 H_2} \frac{\partial H_3}{\delta \beta} \sum_{j=1}^{m} v_j \chi_j \Big\} \Big] \psi_h \; H_3 d\gamma$$

$$+ \int_\gamma \frac{\partial}{\delta \alpha} \Big[G \; H_3 H_2 \Big\{ \frac{H_3}{H_1} \sum_{k=1}^{n} \frac{\partial}{\delta \alpha} \Big(\frac{1}{H_3} w_k \Big) \psi_k$$

$$+ \frac{H_1}{H_3} \sum_{i=1}^{l} \frac{\partial}{\delta \gamma} \Big(\frac{1}{H_1} \phi_i \Big) \; u_i \Big\} \Big] \psi_h \; H_3 d\gamma + \int_\gamma \frac{\partial}{\delta \beta} \Big[G \; H_3 H_1$$

$$\times \Big\{ \frac{H_2}{H_3} \sum_{j=1}^{m} \frac{\partial}{\delta \gamma} \Big(\frac{1}{H_2} \chi_j \Big) \; v_j + \frac{H_3}{H_2} \sum_{k=1}^{n} \frac{\partial}{\delta \beta} \Big(\frac{1}{H_3} w_k \Big) \psi_k \Big] \psi_h H_3 d\gamma$$

$$- \int_{\gamma} \Big[\frac{\lambda}{H_1 H_2 H_3} \Big\{ \sum_{i=1}^{l} \frac{\partial}{\partial \alpha} (H_2 H_3 u_i) \phi_i + \sum_{j=1}^{m} \frac{\partial}{\partial \beta} (H_1 H_3 v_j) \chi_j$$

$$+ \sum_{k=1}^{n} \frac{\partial}{\partial \gamma} (H_2 H_1 \psi_k) w_k \Big\} + 2G \Big\{ \frac{1}{H_1} \sum_{i=1}^{l} \frac{\partial u_i}{\partial \alpha} \phi_i$$

$$+ \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \beta} \sum_{j=1}^{m} v_j \chi_j + \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial \gamma} \sum_{k=1}^{n} w_k \psi_k \Big\} \Big]$$

$$\times H_2 \frac{\partial H_1}{\partial \gamma} \psi_h H_3 \, d\gamma - \int_{\gamma} \Big[\frac{\lambda}{H_1 H_2 H_3} \Big\{ \sum_{i=1}^{l} \frac{\partial}{\partial \alpha} (H_2 H_3 u_i) \phi_i$$

$$+ \sum_{j=1}^{m} \frac{\partial}{\partial \beta} (H_1 H_3 v_j) \chi_j + \sum_{k=1}^{n} \frac{\partial}{\partial \gamma} (H_1 H_2 \psi_k) w_k \Big\}$$

$$+ 2G \Big\{ \frac{1}{H_2} \sum_{j=1}^{m} \frac{\partial v_j}{\partial \beta} \chi_j + \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial \gamma} \sum_{k=1}^{n} w_k \psi_k$$

$$+ \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial \alpha} \sum_{i=1}^{l} u_i \phi_i \Big\} \Big] H_1 \frac{\partial H_2}{\partial \gamma} \psi_h H_3 \, d\gamma$$

$$+ \int_{\gamma} G \Big[\frac{H_2}{H_1} \sum_{k=1}^{n} \frac{\partial}{\partial \alpha} \Big(\frac{1}{H_3} w_k\Big) \psi_k + \frac{H_1}{H_3} \sum_{i=1}^{l} \frac{\partial}{\partial \gamma} \Big(\frac{1}{H_1} \phi_i\Big) u_i \Big]$$

$$\times H_2 \frac{\partial H_3}{\partial \alpha} \psi_h H_3 \, d\gamma + \int_{\gamma} G \Big[\frac{H_2}{H_3} \sum_{j=1}^{m} \frac{\partial}{\partial \gamma} \Big(\Big(\frac{1}{H_2} \chi_j\Big) v_j$$

$$+ \frac{H_3}{H_2} \sum_{k=1}^{n} \frac{\partial}{\partial \beta} \Big(\frac{1}{H_3} w_k\Big) \psi_k \Big] H_1 \frac{\partial H_3}{\partial \beta} \psi_h H_3 \, d\gamma$$

$$+ \int P_{\gamma} H_1 H_2 H_3 \psi_h H_3 \, d\gamma + [\sigma_{\gamma\gamma} H_1 H_2 H_3 \psi_h]_{\gamma} = 0$$

$$(h = 1, 2, \ldots) \qquad (2.13c)$$

Accuracy of the foundation model depends upon the number of terms taken in the series of Eqns. 2.13, and upon a proper choice of the functions $\phi_i(\gamma)$, $\chi_j(\gamma)$ and $\psi_k(\gamma)$ guided by experimental results or from some theoretical considerations.

2.2.3 Choice of Functions $\phi(\gamma)$, $\chi(\gamma)$ and $\psi(\gamma)$

The dimensionless functions $\phi(\gamma)$, $\chi(\gamma)$ and $\psi(\gamma)$ have to be chosen from the physical constraints of the problem. For elastic foundations of finite thickness fixed on a rigid base, the distribution of displacements through the thickness can be assumed to be linearly decreasing, especially for thin layers. In the present investigation, for finite elastic layers, the following functions are used,

$$\phi(\gamma) = \chi(\gamma) = \psi(\gamma) = \frac{H-\gamma}{H} \tag{2.14}$$

where H is the thickness of the foundation layer. If the foundation layer is relatively thick, or of infinite thickness the choice can be made as,

$$\phi(\gamma) = \chi(\gamma) = \psi(\gamma) = \frac{\operatorname{Sinh} C(H-\gamma)}{\operatorname{Sinh} CH} \tag{2.15}$$

where C is a coefficient depending on the elastic properties of the foundation and defines the rate of decrease of displacement with depth. Another possible choice for foundation extending to infinity can be,

$$\phi(\gamma) = \chi(\gamma) = \psi(\gamma) = e^{-c\gamma} \tag{2.16}$$

where c is constant coefficient depending on the elastic properties of the foundation. Many such expressions can be selected depending upon experimental or theoretical considerations for the given nature of the foundation.

2.3 SHELL EQUATIONS:

Let α, β and γ be the three curvilinear orthogonal coordinates as shown in Figs. (2.2). Forces and moments acting on the shell element are shown in Figs. (2.2a) and (2.2b) respectively.

The equations of equilibrium can be written as,

$$\frac{\partial}{\partial\alpha}(BN_1) - N_2\frac{\partial B}{\partial\alpha} + \frac{\partial}{\partial\beta}(AS_2) + S_1\frac{\partial A}{\partial\beta} + \\ + AB\,k_1Q_1 + ABX = 0 \tag{2.17a}$$

$$\frac{\partial}{\partial\beta}(AN_2) - N_1\frac{\partial A}{\partial\beta} + \frac{\partial}{\partial\alpha}(BS_1) + S_2\frac{\partial B}{\partial\alpha} \\ + ABk_2\,Q_2 + ABY = 0 \tag{2.17b}$$

$$-(k_1N_1 + k_2N_2) + \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BQ_1) + \frac{\partial}{\partial\beta}(AQ_2)\right] + Z = 0 \tag{2.17c}$$

$$\frac{\partial}{\partial\alpha}(BM_{12}) + M_{21}\frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta}(AM_2) + M_1\frac{\partial A}{\partial\beta} - ABQ_2 = 0 \tag{2.17d}$$

$$\frac{\partial}{\partial\beta}(AM_{21}) + M_{12}\frac{\partial A}{\partial\beta} - \frac{\partial}{\partial\alpha}(BM_1) + M_2\frac{\partial B}{\partial\alpha} - ABQ_1 = 0 \tag{2.17e}$$

where $A = A(\alpha, \beta)$ and $B = B(\alpha, \beta)$ are the coefficients of the first quadratic form of the surface in orthogonal coordinates α and β,

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 \qquad (2.18)$$

k_1, k_2 are the curvature, X,Y are the projections of the load along the positive tangents to the coordinate lines α and β respectively, Z is the projection of the load along the exterior normal [Fig.. (2.2a)].

N_1, S_1 and N_2, S_2 are the normal and the shearing forces acting on the area of a normal section of the body of a shell along the lines α = constant and β = constant respectively. M_1, M_{12}, Q_1 and M_2, M_{21}, Q_2 are the bending moments, twisting moments and transverse forces towards the positive tangent to the lines α = constant and β = constant respectively.

The first three of the Eqns. (2.17) express the condition that the sum of the projection of all forces acting on the isolated element of the shell in three mutually orthogonal directions, is equal to zero. The fourth and fifth equations are obtained from the condition that the moments of all forces acting on the isolated element of the shell about the two mutually perpendicular axes lying in a plane tangent to the middle surface are equal to zero.

The sixth equation of equilibrium obtained by equating the sum of the moments about an axis directed along the normal to the middle surface to zero, will have the form,

$$S_1 - S_2 + k_1 M_{12} - k_2 M_{21} = 0 \tag{2.19}$$

Eqn. (2.19) can be shown to be satisfied identically with the use of force-deformation relation mentioned below.

Force-deformation relations can be written as,

$$N_1 = \frac{E_s h}{(1-\mu_s^2)} [\varepsilon_1 + \mu_s \varepsilon_2 - \frac{h^2}{12} (k_1 - k_2) \xi_1] \tag{2.20a}$$

$$S_1 = \frac{E_s h}{2(1+\mu_s)} [\omega \quad \frac{h^2}{12} (k_1 - k_2) \tau] \tag{2.20b}$$

$$M_1 = - \frac{E_s h^3}{12(1-\mu_s^2)} [\xi_1 + \mu_s \xi_2 + k_2 (\varepsilon_1 + \mu_s \varepsilon_2)] \tag{2.20c}$$

$$M_{12} = \frac{E_s h^3}{24(1+\mu_s)} [2\tau + k_2 \omega] \tag{2.20d}$$

$$N_2 = \frac{E_s h}{(1-\mu_s^2)} [\varepsilon_2 + \mu_s \varepsilon_1 + \frac{h^2}{12} (k_1 - k_2) \xi_2] \tag{2.20e}$$

$$S_2 = \frac{E_s h}{2(1+\mu_s)} [\omega + \frac{h^2}{12} (k_1 - k_2) \tau] \tag{2.20f}$$

$$M_2 = \frac{- E_s h^3}{12(1-\mu_s^2)} [\xi_2 + \mu_s \xi_1 + k_1 (\varepsilon_2 + \mu_s \varepsilon_1)] \tag{2.20g}$$

$$M_{21} = \frac{E_s h^3}{24(1+\mu_s)} \left[2\tau + k_1 \omega \right] \tag{2.20h}$$

also,

$$Q_1 = \frac{1}{AB} \left[\frac{\partial}{\partial\beta} (A\, M_{21}) + M_{12} \frac{\partial A}{\partial\beta} - \frac{\partial}{\partial\alpha} (BM_1) + M_2 \frac{\partial B}{\partial\alpha} \right] \tag{2.20i}$$

$$Q_2 = \frac{1}{AB} \left[\frac{\partial}{\partial\alpha} (AM_{12}) + M_{21} \frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta} (AM_2) + M_1 \frac{\partial A}{\partial\beta} \right] \tag{2.20j}$$

where strain-displacement relations are given as,

$$\varepsilon_1 = \frac{1}{A} \frac{\partial u_s}{\partial\alpha} + \frac{1}{AB} \frac{\partial A}{\partial\beta} v_s + k_1 w_s \tag{2.21a}$$

$$\varepsilon_2 = \frac{1}{AB} \frac{\partial B}{\partial\alpha} u_s + \frac{1}{B} \frac{\partial v_s}{\partial\beta} + k_2 w_s \tag{2.21b}$$

$$\omega = \frac{A}{B} \frac{\partial}{\partial\beta} \left(\frac{u_s}{A}\right) + \frac{B}{A} \frac{\partial}{\partial\alpha} \left(\frac{v_s}{B}\right) \tag{2.21c}$$

$$\xi_1 = \frac{\partial k_1}{\partial\alpha} \frac{u_s}{A} + \frac{\partial k_1}{\partial\beta} \frac{v_s}{B} - k_1^2 w_s - \frac{1}{A} \frac{\partial}{\partial\alpha} \left(\frac{1}{A} \frac{\partial w_s}{\partial\alpha}\right)$$

$$- \frac{1}{AB^2} \frac{\partial A}{\partial\beta} \frac{\partial w_s}{\partial\beta} \tag{2.21d}$$

$$\xi_2 = \frac{\partial k_2}{\partial\alpha} \frac{u_s}{A} + \frac{\partial k_2}{\partial\beta} \frac{v_s}{B} - k_2^2 w_s - \frac{1}{B} \frac{\partial}{\partial\beta} \left(\frac{1}{B} \frac{\partial w_s}{\partial\beta}\right)$$

$$- \frac{1}{A^2 B} \frac{\partial B}{\partial\alpha} \frac{\partial w_s}{\partial\alpha} \tag{2.21e}$$

$$\tau = \frac{k_1 - k_2}{2} \left[\frac{A}{B} \frac{\partial}{\partial\beta} \left(\frac{u_s}{A}\right) - \frac{B}{A} \frac{\partial}{\partial\alpha} \left(\frac{v_s}{B}\right) \right]$$

$$- \frac{2}{AB} \left[\frac{\partial^2 w_s}{\partial\alpha\partial\beta} - \frac{1}{A} \frac{\partial A}{\partial\beta} \frac{\partial w_s}{\partial\alpha} - \frac{1}{B} \frac{\partial B}{\partial\alpha} \frac{\partial w_s}{\partial\beta} \right] \tag{2.21f}$$

where u_s, v_s and w_s are the displacements in α, β and γ directions respectively, ε_1, ε_2 are the normal strains in α, β directions, ω is the shear strain characterised by the change in right angle between the coordinates α = constant and β = constant, ξ_1, ξ_2 are associated with bending deformation in α and β directions and τ represents relative twisting deformation.

In case of thin shells the moment terms containing the factors k_1, k_2, $k_1 . k_2$ and the derivatives of these quantities and associated with the tangent displacements u_s and v_s have very little effect on the internal stresses and deformations of shells. The smaller the relative thickness of a shell, which is determined by the quantity h/R_{min} where R_{min} is the smallest radius of curvature and h is the thickness of the shell, the smaller the role of these terms.

In the case of thin shells in which $h/R_{min} \leq 1/30$ all terms in basic equations which contain the quantities $h^2k_1/12$, $h^2k_2/12$ and partial derivatives of these quantities can be neglected without appreciable error.

The formulas for deformationx component of a shell [Eqns. (2.21)] reduces to

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u_s}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v_s + k_1 w_s \tag{2.22a}$$

$$\varepsilon_2 = \frac{1}{B}\frac{\partial v_s}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u_s + k_2 w_s \qquad (2.22b)$$

$$\omega = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u_s}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v_s}{B}\right) \qquad (2.22c)$$

$$\xi_1 = -\frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w_s}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w_s}{\partial \beta} \qquad (2.22d)$$

$$\xi_2 = -\frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w_s}{\partial \beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial \alpha}\frac{\partial w_s}{\partial \alpha} \qquad (2.22e)$$

$$\tau = -\frac{1}{AB}\left[\frac{\partial^2 w_s}{\partial \alpha \partial \beta} - \frac{1}{A}\frac{\partial A}{\partial \beta}\frac{\partial w_s}{\partial \alpha} - \frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{\partial w_s}{\partial \beta}\right] \qquad (2.22f)$$

Eqns. (2.20) acquire the following simple form when small terms proportional to the quantities $h^3k_1/12$, $h^3k_2/12$ are neglected.

$$N_1 = \frac{E_s h}{(1-\mu_s{}^2)}\left[\varepsilon_1 + \mu_s\ \varepsilon_2\right] \qquad (2.23a)$$

$$N_2 = \frac{E_s h}{(1-\mu_s{}^2)}\left[\varepsilon_2 + \mu_s\ \varepsilon_1\right] \qquad (2.23b)$$

$$S = S_1 = S_2 = \frac{E_s h}{2(1+\mu_s)}\,\omega \qquad (2.23c)$$

$$M_1 = -\frac{E_s h^3}{12(1-\mu_s{}^2)}\left[\xi_1 + \mu_s\,\xi_2\right] \qquad (2.23d)$$

$$M_2 = -\frac{E_s h^3}{12(1-\mu_s{}^2)}\left[\xi_2 + \mu_s\,\xi_1\right] \qquad (2.23e)$$

$$M_{12} = M_{21} = \frac{G_s h^3}{12(1+\mu_s)} \tau \qquad (2.23f)$$

$$Q_1 = \frac{1}{AB} \left[\frac{\partial}{\partial \beta} (AM_{12}) + M_{12} \frac{\partial A}{\partial \beta} - \frac{\partial}{\partial \alpha} (BM_1) + M_2 \frac{\partial B}{\partial \alpha} \right] \qquad (2.23g)$$

$$Q_2 = \frac{1}{AB} \left[\frac{\partial}{\partial \alpha} (BM_{12}) + M_{12} \frac{\partial B}{\partial \alpha} - \frac{\partial}{\partial \beta} (AM_2) + M_1 \frac{\partial A}{\partial \beta} \right] \qquad (2.23h)$$

Equations of equilibrium [Eqns. (2.17)] for thin shell can be rewritten as,

$$\frac{\partial}{\partial \alpha} (BN_1) - N_2 \frac{\partial B}{\partial \alpha} + \frac{\partial}{\partial \beta} (AS) + S \frac{\partial A}{\partial \beta} + ABX = 0 \qquad (2.24a)$$

$$\frac{\partial}{\partial \beta} (AN_2) - N_1 \frac{\partial A}{\partial \beta} + \frac{\partial}{\partial \alpha} (BS) + S \frac{\partial B}{\partial \alpha} + ABY = 0 \qquad (2.24b)$$

$$-(k_1 N_1 + k_2 N_2) + \frac{1}{AB} \left[\frac{\partial}{\partial \alpha} (BQ_1) + \frac{\partial}{\partial \beta} (AQ_2) \right] + Z = 0 \qquad (2.24c)$$

$$\frac{\partial}{\partial \alpha} (BM_{12}) + M_{12} \frac{\partial B}{\partial \alpha} - \frac{\partial}{\partial \beta} (AM_2) + M_1 \frac{\partial A}{\partial \beta} - ABQ_2 = 0 \qquad (2.24d)$$

$$\frac{\partial}{\partial \beta} (AM_{12}) + M_{12} \frac{\partial A}{\partial \beta} - \frac{\partial}{\partial \alpha} (BM_1) + M_2 \frac{\partial B}{\partial \alpha} - ABQ_1 = 0 \qquad (2.24e)$$

In the first two equations of (2.21), the quantities $k_1 Q_1$, $k_2 Q_2$ which result from the moments are proportional to curvatures k_1, k_2 of the shell, are neglected due to their smallness.

By excluding from the Eqns. (2.24) the transverse forces Q_1, Q_2 and expressing the stress resultants N_1, N_2, S and the moments M_1, M_2, M_{12} by means of deformations ε_1, ε_2, ω, ξ_1, ξ_2 and τ [Eqns. (2.23)] and these deformations expressed in terms of the displacements u_s, v_s and w_s [Eqns. (2.22)], and making use of compatibility conditions, the equations of equilibrium can be written as [28],

$$\frac{1}{A}\frac{\partial\Delta_o}{\partial\alpha} - (1-\mu_s)\frac{1}{B}\frac{\partial\xi_o}{\partial\beta} + (1-\mu_s)\left(Ku_s - \frac{k_2}{A}\frac{\partial w_s}{\partial\alpha}\right) = -\frac{(1-\mu_s^2)}{E_s h}\lambda \tag{2.25a}$$

$$\frac{1}{B}\frac{\partial\Delta_o}{\partial\beta} + (1-\mu_s)\frac{1}{A}\frac{\partial\xi_o}{\partial\alpha} + (1-\mu_s)\left(Kv_s - \frac{k_1}{B}\frac{\partial w_s}{\partial\beta}\right) = -\frac{(1-\mu_s^2)}{E_s h}Y \tag{2.25b}$$

and,

$$-(k_1+k_2)\Delta_o + (1-\mu_s)\frac{1}{AB}\left[2AB\ Kw_s + \frac{\partial}{\partial\alpha}(Bk_2\ u_s) + \frac{\partial}{\partial\beta}(Ak_1\ v_s)\right] - \frac{h^2}{12}\nabla^2(k_1^2 + k_2^2)\ w_s - \frac{h^2}{12}\nabla^2\nabla^2 w_s = -\frac{(1-\mu_s^2)}{E_s h} Z \tag{2.25c}$$

where,

$$\Delta_o = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(Bu_s) + \frac{\partial}{\partial\beta}(Av_s)\right] + (k_1 + k_2)\ w_s \tag{2.25d}$$

$$\xi_o = \frac{1}{2AB}\left[\frac{\partial}{\partial\alpha}(Bv_s) - \frac{\partial}{\partial\beta}(Au_s)\right] \qquad (2.25e)$$

$$\nabla^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right] \qquad (2.25f)$$

and, $$K = k_1 \cdot k_2 \qquad (2.25g)$$

Eqns. (2.25) represent thin shells in terms of displacements u_s, v_s and w_s. In addition to the assumptions made for thin shells, the general technical theory of thin shells is based on another assumption that the middle surface of the shell possesses metric of the Euclidean geometry. The geometric sense of this assumption, which is very important in shallow shells, is that the expressions for the first quadratic of the surface,

$$ds^2 = A^2\, d\alpha^2 + B^2\, d\beta^2 \qquad (2.26)$$

irrespective of the Gaussian curvature ($K = k_1, k_2$) of this surface is identified with the analogous expression for the first quadratic form on a plane.

The Gauss equation,

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = -KAB \qquad (2.27)$$

which holds good for any surface, is replaced by the following simpler equation in the case of a shallow shell outlined by a comparatively small part of the surface; on the strength of

the hypothesis it can be written,

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}\right) = 0 \qquad (2.28)$$

irrespective of the value of the Gaussian curvature. This means that if one superimposes on a small portion of the surface two families of lines corresponding to any system of orthogonal coordinates then these lines will not differ in any respect from the corresponding coordinate lines on the plane over which the given surface rises.

2.4 NUMERICAL METHOD

Complex structural systems necessitate the use of approximate methods for static and dynamic analyses. With the advent of the high speed computers, the method of step-wise numerical integration has proved to be a powerful and accurate method for a large classes of problems.

There are number of methods for the numerical integration of a system of linear ordinary differential equations. Many of these methods can be grouped under the class of predictor-corrector methods or Runge-Kutta methods. Since boundary value problems are of major concern to us, the above methods which are suitable for initial value problems, should be modified to suit the boundary value problems. Such methods are called shooting methods [30]. These

methods are capable of solving both linear and nonlinear boundary value problems. Shooting methods are also called segmentation method and multisegment method. In these methods the differential equations of m th order are first converted into a set of m first order differential equations which are two point boundary value problems.

In case of boundary value problems since only part of the boundary conditions at each boundary are prescribed, a possible method of using step wise integration would be to start with a set of trial values for the missing initial conditions at the starting point. Then, the calculated terminal conditions can be compared with the prescribed conditions at that point, and further trial can be made if there is no agreement between them. However, such a hit-and miss method is not suitable for machine computations. Hence a systematic method of determining the missing initial conditions is to be used. Once the boundary value problem is converted into a initial value problem, the direct integration approach can be used to find the solution.

While the direct integration method has a definite advantage, it also has serious disadvantage; that is when the length of the shell is large, a loss of accuracy invariably results. This loss of accuracy does not result from accumulative errors in integration but it is caused by

the subtraction of almost equal numbers in the process of determination of the unknown boundary values. That is, for every set of geometric and material parameters of the shell there is a critical length beyond which the solution looses all accuracy. The advantage of finite difference approach over direct integration is that it can avoid such loss of accuracy. It is concluded [29] that if the solutions of the system of algebric equations, which results from the finite difference equation is obtained by means of Gaussian elemination, then no loss of accuracy is experienced if the length of the shell is increased.

The method used here is given by Kalnins [29]. This method has definite advantage over the finite difference approach. The main advantages are (a) it can be applied conveniently to a large system of first order differential equations and (b) it permits selection of an optimum step size of integration at each step according to the desired accuracy of the solution.

The method used here can be divided into two parts (a) direct integration of (m + 1) initial value problems over preselected segments of the total interval and (b) the use of Gaussian elemination for the solution of resulting system of matrix equations.

The initial value problems are defined over segments of the total interval, the length of which are within the range of applicability of the direct integration approach. After the initial value problems are integrated over these segments, continuity conditions on all variables are written at the end points of the segments and they constitute a simultaneous system of linear matrix equations. This system of matrix equations is then solved directly by means of Gaussian elimination. The result is that direct integration approach is applied and there is no loss of accuracy because the lengths of the segments are selected in such a way that the solutions of the initial value problems are kept sufficiently small.

2.4.1 Reduction of Boundary Value Problem to Initial Value Problem:

Two point boundary value problem governed by differential equation,

$$\frac{dy(x)}{dx} = A(x)\,y(x) + B(x) \tag{2.29}$$

is reduced to a series of initial value problems. In (2.29) $y(x)$ is an $(m,1)$ matrix which represents m unknown functions; x is the independent variable; $A(x)$ denotes the (m, m) coefficient matrix and $B(x)$ is the $(m, 1)$ matrix of non-homogeneous terms. The object is to determine $y(x)$ in the

interval $a \leq x \leq b$ subjected to m boundary conditions in terms of linear combination of y(a) and y(b) in the form,

$$F_a \, y(a) + F_b \, y(b) = G \qquad (2.30)$$

where F_a, F_b are (m, m) matrices and G is an (m, 1) matrix which are known from the statement of the boundary conditions of the problem.

Let the complete solution of (2.29) be written as,

$$y(x) = Y(x).C + Z(x) \qquad (2.31)$$

where the (m, 1) matrix C represents m arbitrary constants and Y(x) is an (m, m) and Z(x) an (m, 1) matrix which are defined as the homogeneous and particular solutions of (2.29) in the form,

$$\frac{dY(x)}{dx} = A(x) \, Y(x) \qquad (2.32a)$$

and $$\frac{dZ(x)}{dx} = A(x).Z(x) + B(x) \qquad (2.32b)$$

The initial conditions for determining Y(x) and Z(x) are

$$Y(a) = I \qquad (2.33a)$$

and $$Z(a) = 0 \qquad (2.33b)$$

where I is a unit matrix.

Evaluation of (2.31) at $x = a$ leads at once, in view of Eqns. (2.33a), (2.33b) to $C = y(a)$ and then (2.31) at $x = b$ can be written as,

$$y(b) = Y(b)\, y(a) + Z(b) \tag{2.34}$$

Together with (2.30), equation (2.34) constitutes a system of 2m linear algebraic equations from which the 2m unknowns $y(a)$ and $y(b)$ are determined. Once $y(a)$ is known, the solution at any value of x is obtained from (2.31) provided $Y(x)$ and $Z(x)$ at that particular x are stored. This completes the reduction of a two point boundary value problem defined by (2.29) to (m +1) initial value problems given by Eqns. (2.32) and (2.33).

2.4.2 Multisegment Method of Integration:

A differential equation system of even order, say m, with m/2 known boundary conditions at either boundary will be considered. However, the method is equally applicable to a more general case. Let the shell be devided into M-segments, denoted by Si, $i = 1,2,\ldots M$, of arbitrary length. Let the coordinates of the ends of the segments be $x = x_i$, where the left hand edge of the shell is x_1 and right hand edge is at $x = x_{M+1}$. Then the solution of equation (2.29) following the analogy of equation (2.30) can be written as,

$$y(x) = Y_i(x)\, y(x_i) + Z_i(x) \qquad (2.35)$$

$$i = 1,2,\ldots M$$

where $Y_i(x)$ denotes the fundamental matrix solution for the segment S_i and Z_i is the corresponding particular integral. Y_i and Z_i are defined numerically at finite number of points.

The intersegment continuity condition can be written as,

$$y(x_{i+1}) = Y_i(x_{i+1})\, y(x_i) + Z_i(x_{i+1}) \qquad (2.36)$$

$$i = 1,2,\ldots M$$

The continuity condition between two successive segments are written in a matrix, form for all the segments.

Forr example, it can be written xthat,

$$y_2 = Y_1 y_1 + Z_1$$

$$y_3 = Y_2 y_2 + Z_2$$

$$= Y_2(Y_1 y_1 + Z_1) + Z_2$$

$$\vdots$$

$$y_{M+1} = Y_M\, y_M + Z_M$$

$$= Y_M\, Y_{M-1}\, Y_{M-2} \;\ldots\; Y_2 Y_1 y_1 + Y_M\, Y_{M-1} \ldots Y_2 Z_1$$

$$+ Y_M\, Y_{M-1} \ldots Y_3 Z_2 + \ldots + Y_M\, Z_{M-1} + Z_M$$

or
$$y_1 = [Y_M\, Y_{M-1} \ldots Y_2 Y_1]^{-1}\, y_{M+1} - Y_1^{-1} Z_1 - Y_1^{-1} Y_2^{-1} Z_2$$

$$- \ldots\ldots - Y_1^{-1}\, Y_2^{-1} \ldots Y_M^{-1}\, Z_M \qquad (2.37)$$

Once y_1 is known in terms of y_{M+1}, the resulting algebraic equations can be solved using Gauss elemination. The matrix can be written as,

$$\begin{bmatrix} \underset{(m\times m)}{-I} & \underset{(m\times m)}{\bar{Y}} \\ \underset{(\frac{m}{2}\times m)}{C_1} & - \\ & \underset{(\frac{m}{2}\times m)}{C_2} \end{bmatrix} \begin{Bmatrix} \underset{(m\times 1)}{y_1} \\ \\ \underset{(m\times 1)}{y_{M+1}} \end{Bmatrix} = \begin{Bmatrix} \underset{(m\times 1)}{Q_Q^{(1)}} \\ \\ \underset{(m+1)}{Q_Q^{(2)}} \end{Bmatrix} \tag{2.38}$$

where,

$$\underset{(m\times m)}{\bar{Y}} = [Y_m \cdot Y_{m-1} \ldots\ldots Y_2\, Y_1]^{-1} \tag{2.39a}$$

and

$$\underset{(m\times 1)}{Q_Q^{(1)}} = Y_1^{-1} Z_1 + Y_1^{-1} Y_2^{-1} Z_2 + \ldots$$
$$\ldots + Y_1^{-1} Y_2^{-1} \ldots Y_M^{-1} Z_M \tag{2.39b}$$

and, $Q_Q^{(2)}$ is the R.H.S. of the boundary conditions at x_1 and x_{M+1}.

This gives all the functional values at x_1 and x_{M+1}. Using Runge-Kutta method of integration, values of functions for all x in $a \leq x \leq b$ can be obtained.

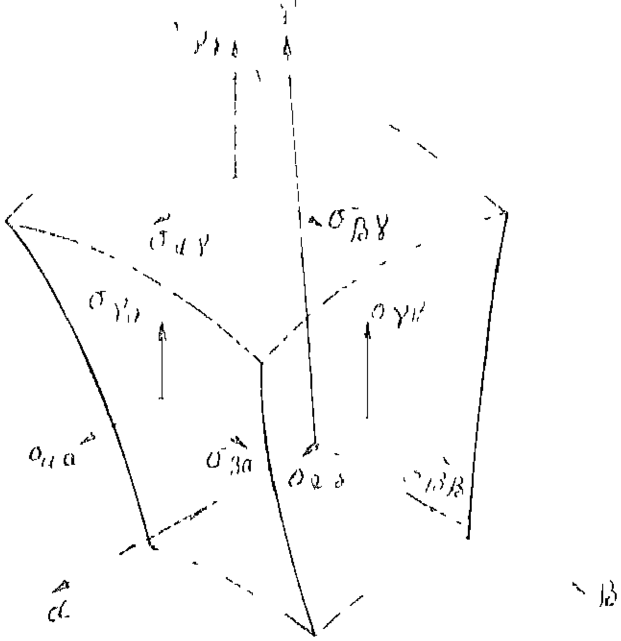
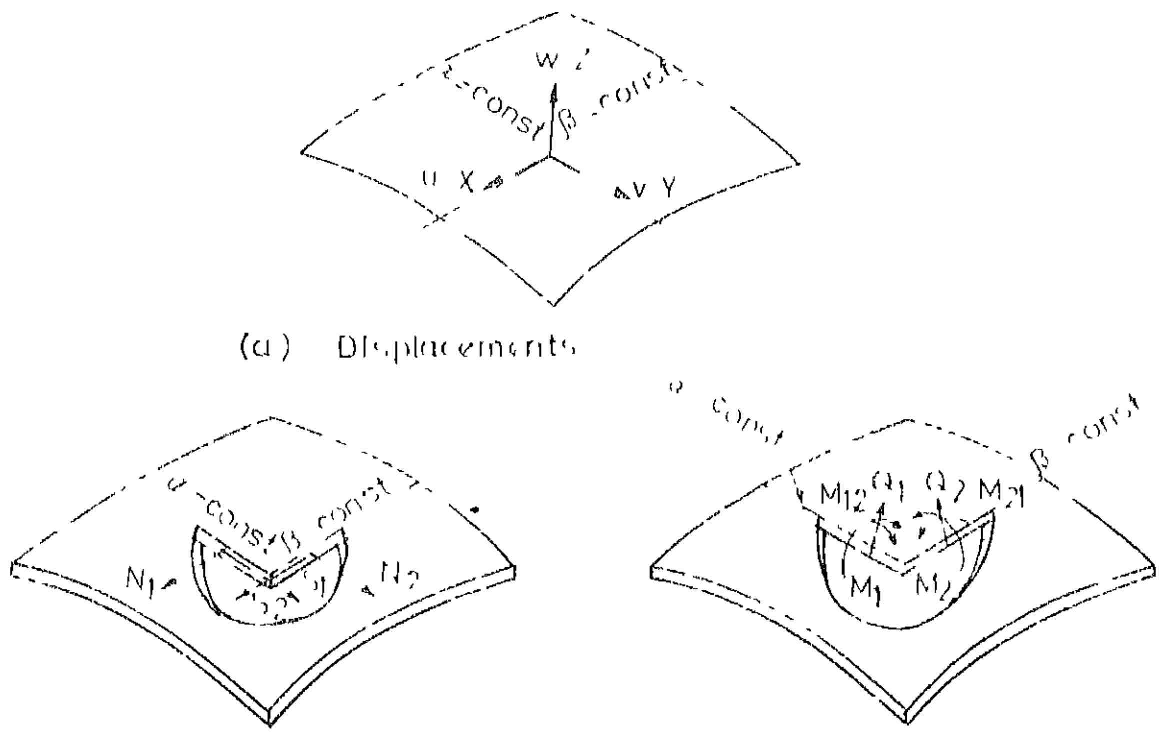


FIG. 2.[illegible] FOUNDATION ELEMENT IN ORTHOGONAL CURVILINEAR CO-ORDINATES



(a) Displacements

(b) In plane forces

(c) Moments & transverse forces

FIG. 2.[illegible] SHELL ELEMENT IN ORTHOGONAL CURVILINEAR CO-ORDINATES

CHAPTER III

AXISYMMETRIC PROBLEMS

3.1 GENERAL:

Equations for foundation model and shells developed in Chapter II are specialised for thin shallow spherical shells on elastic foundations and thin shallow conical shells on elastic foundations, applicable to axisymmetric problems.

Shallow spherical and conical shells subjected to normal load transferred by the column have been solved for fixed, simply supported and free types of boundary conditions. The foundation thickness has been considered as finite in all the examples that have been solved. Foundation of infinite thickness, layered elastic foundation could also be considered without any difficulty. Results have been presented in non-dimensional form so that they can be used for design purposes.

In case of spherical shells two parameters, which accounts for thinness of the shell (h/R) and shallowness of the shell (R/a) have been considered. Results have been presented for different values of these parameters. In case

of conical shells results are presented for different values of Θ, where Θ is the angle made by generators and axis of the shell. Θ accounts for the shallowness of the conical shell.

3.2 EQUATIONS FOR THIN SHALLOW SHELLS FOR AXISYMMETRIC PROBLEMS:

3.2.1 Spherical Shell:

In case of shallow shell, polar coordinate system can be taken as,

$$ds^2 = dr^2 + r^2\, d\beta^2 \tag{3.1}$$

Comparing Eqn. (3.1) with the first quadratic equation of surface in curvilinear coordinates, Eqn. (2.24), it can be seen that,

$$A = 1,\ B = r,\ d\alpha = dr \text{ and } d\beta = d\beta \tag{3.2}$$

For spherical shells both the curvatures are equal, [Fig. (3.1a)]

$$k_1 = k_2 = k = \frac{1}{R} \tag{3.3}$$

This gives Gaussian curvature as

$$K = k_1\, k_2 = \frac{1}{R^2} \tag{3.4}$$

In axisymmetric problems, the displacement functions are independent of β and the displacement function v_s in the circumferential direction is zero.

$$u_s = u_s(r) \tag{3.5a}$$

$$w_s = w_s(r) \tag{3.5b}$$

Substituting Eqns. (3.2), (3.3), (3.4) and (3.5) in the shell equations in curvilinear coordinates [Eqn. (2.25)], the resulting expressions are,

$$\frac{d^2u_s}{dr^2} + \frac{1}{r}\frac{du_s}{dr} - \frac{u_s}{r^2} + \frac{(1-\mu_s)}{R^2}u_s + \frac{(1+\mu_s)}{R}\frac{dw_s}{dr}$$

$$= -\frac{(1-\mu_s^2)}{E_s h}X \qquad (3.6a)$$

$$-\frac{h^2}{12}\left[\frac{d^4w_s}{dr^4} + \frac{2}{r}\frac{d^3w_s}{dr^3} - \frac{1}{r^2}\frac{d^2w_s}{dr^2} + \frac{2}{R^2}\frac{d^2w_s}{dr^2} + \frac{1}{r^3}\frac{dw_s}{dr}\right.$$

$$\left. + \frac{2}{rR^2}\frac{dw_s}{dr}\right] - \frac{2(1+\mu_s)}{R^2}w_s - \frac{(1+\mu_s)}{R}\frac{du_s}{dr} - \frac{(1+\mu_s)}{Rr}u_s$$

$$= -\frac{(1-\mu_s^2)}{E_s h}Z \qquad (3.6b)$$

Equations (3.6a) and (3.6b) represent thin shallow spherical shell applicable only to axisymmetric problems.

3.2.2 Conical Shells:

Polar coordinate system can also be taken in case of conical shells, hence Eqns. (3.1) and (3.2) are also valid in case of shallow conical shells. For conical shells [Fig. 3.1b] radius of curvature R_1 is infinite and

$$R_2 = r/\cos\theta \qquad (3.7)$$

$$\text{also} \quad K = 0 \qquad (3.8)$$

For axisymmetric problems,

$$u_s = u_s(r) \qquad (3.9a)$$

$$w_s = w_s(r) \qquad (3.9b)$$

and

$$v_s = 0 \qquad (3.9c)$$

Substituting Eqns. (3.2), (3.7), (3.8) and (3.9) in the shell Eqns. (2.25), the resulting expressions are,

$$\frac{d^2u_s}{dr^2} + \frac{1}{r}\frac{du_s}{dr} - \frac{u_s}{r^2} - \frac{\cos\theta}{r^2} w_s + \frac{\mu_s \cos\theta}{r}\frac{dw_s}{dr} = -\frac{(1-\mu_s^2)}{E_s h} X \qquad (3.10a)$$

$$-\frac{h^2}{12}\left[\frac{d^4w_s}{dr^4} + \frac{2}{r}\frac{d^3w_s}{dr^3} - \frac{(1-\cos^2\theta)}{r^2}\frac{d^2w_s}{dr^2} + \frac{(1-3\cos^2\theta)}{r^3}\frac{dw_s}{dr} + \frac{4\cos^2\theta}{r^4} w_s\right] - \frac{\cos^2\theta}{r^2} w_s$$

$$-\frac{\mu_s \cos\theta}{r}\frac{du_s}{dr} - \frac{\cos\theta}{r^2} u_s = -\frac{(1-\mu_s^2)}{E_s h} Z \qquad (3.10b)$$

Eqns. (3.10) represent thin shallow conical shell applicable to axisymmetric problems only.

3.3 EQUATIONS FOR ELASTIC FOUNDATIONS APPLICABLE TO AXISYMMETRIC PROBLEMS:

Equations for elastic foundation are developed in Chapter II. These foundation equations are particularised to

suit for shallow spherical shell and shallow conical shell. The first quadratic equation is taken as,

$$ds^2 = dr^2 + r^2 d\beta^2 + d\gamma^2 \tag{3.11}$$

Comparing Eqn. (3.11) with the first quadratic equation in curvilinear coordinates [Eqn.(2.2)], it can be seen that,

$$d\alpha = dr,\ d\beta = d\beta,\ d\gamma = d\gamma,\ H_1 = 1,\ H_2 = r \text{ and } H_3 = 1 \tag{3.12}$$

Substituting Eqns. (3.12) in foundation equations in curvilinear coordinates Eqns. (2.13), the resulting expressions are,

$$\sum_{i=1}^{l} a_{if} \frac{d^2u_i}{dr^2} + \sum_{i=1}^{l} \frac{a_{if}}{r} \frac{du_i}{dr} - \sum_{i=1}^{l} \frac{a_{if}}{r^2} u_i - \sum_{i=1}^{l} b_{if} u_i + \sum_{k=1}^{n} d_{kf} \frac{dw_k}{dr} - \sum_{k=1}^{n} c_{kf} \frac{dw_k}{dr} = - \bar{F}_{rf}$$

$$(f = 1, 2, \ldots \quad) \tag{3.13a}$$

$$\sum_{k=1}^{n} r_{kh} \frac{d^2w_k}{dr^2} + \sum_{k=1}^{n} \frac{r_{kh}}{r} \frac{dw_k}{dr} - \sum_{k=1}^{n} s_{kh} w_k - \sum_{i=1}^{l} d_{ih} \frac{du_i}{dr} + \sum_{i=1}^{l} c_{ih} \frac{du_i}{dr} + \sum_{i=1}^{l} \frac{c_{ih}}{r} u_i - \sum_{i=1}^{l} \frac{d_{ih}}{r} u_i = - \bar{P}_{\gamma h} \tag{3.13b}$$

$$(h = 1, 2, \ldots \quad)$$

where,

$$a_{if} = \int_0^H \frac{E_0}{(1-\mu_0^2)} \phi_i \phi_f \, d\gamma \tag{3.14a}$$

$$b_{lf} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \phi_l' \phi_f' \, d\gamma \qquad (3.14b)$$

$$c_{kf} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \psi_k . \phi_f' \, d\gamma \qquad (3.14c)$$

$$c_{lh} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \phi_l' \psi_h \, d\gamma \qquad (3.14d)$$

$$d_{kf} = \int_0^H \frac{E_o \mu_o}{(1 - \mu_o^2)} \psi_k' \phi_f \, d\gamma \qquad (3.14e)$$

$$d_{lh} = \int_0^H \frac{E_o \mu_o}{(1 - \mu_o^2)} \psi_h' \phi_l \, d\gamma \qquad (3.14f)$$

$$r_{kh} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \psi_k \psi_h \, d\gamma \qquad (3.14g)$$

$$s_{kh} = \int_0^H \frac{E_o}{(1 - \mu_o^2)} \psi_k' \psi_h' \, d\gamma \qquad (3.14h)$$

$$\overline{P}_{rf} = \int_0^H P_r (r, \gamma) \phi_g . d\gamma + [\sigma_{r\gamma} \phi_f]_0^H \qquad (3.14i)$$

$$\overline{P}_{\gamma h} = \int_0^H P_\gamma (r, \gamma) \psi_h \, d\gamma + [\sigma_{\gamma\gamma} \psi_h]_0^H \qquad (3.14j)$$

and,

$$E_o = \frac{E}{(1 - \mu^2)} \qquad (3.15a)$$

$$\mu_o = \frac{\mu}{1 - \mu} \qquad (3.15b)$$

$$a_{11} = \int_0^H \frac{E_o}{(1 - \mu_o^2)} \phi_1^2 \, d\gamma \qquad (3.18a)$$

$$b_{11} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \phi_1'^2 \, d\gamma \qquad (3.18b)$$

$$c_{11} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \psi_1 \phi_1' \, d\gamma \qquad (3.18c)$$

$$d_{11} = \int_0^H \frac{E_o \mu_o}{(1 - \mu_o^2)} \psi_1' \phi_1 \, d\gamma \qquad (3.18d)$$

$$r_{11} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \psi_1^2 \, d\gamma \qquad (3.18e)$$

$$s_{11} = \int_0^H \frac{E_o}{(1 - \mu_o^2)} \psi_1'^2 \, d\gamma \qquad (3.18f)$$

$$\overline{P}_{r1} = \int_0^H P_r (r, \gamma) \phi_1 (\gamma) \, d\gamma + [\sigma_{r\gamma} \phi_1(\gamma)]_0^H \qquad (3.18g)$$

$$\overline{P}_{\gamma 1} = \int_0^H P_\gamma (r, \gamma) \psi_1 (\gamma) \, d\gamma + [\sigma_{\gamma\gamma} \psi_1(\gamma)]_0^H \qquad (3.18h)$$

In the case of externally applied loads on the surface of the body (along $\gamma = 0$), first part of the right hand side of Eqns. (3.18g), (3.18h) have to be taken as Stieltjes Integrals, as given below;

$$\int_0^H P_r (r, \gamma) \phi_1(\gamma) \, d\gamma = P_r(r) \phi_1(0) + \int_\gamma P_r (r, \gamma) \phi_1 (\gamma) \, d\gamma \qquad (3.19a)$$

$$\int_0^H P_\gamma\ (r,\ \gamma)\,\psi_1\ (\gamma)\ d\gamma\ =\ P_\gamma(r)\,\psi_1(0)\ +$$

$$+\int_\gamma P_\gamma(P,\gamma)\,\psi_1(\gamma)\ d\gamma \quad (3.19b)$$

where $P_r(r)$ and $P_\gamma(r)$ are the externally applied surface loads acting along r and γ directions respectively. The body force components $P_r\ (r,\ \gamma)$ and $P_\gamma\ (r,\ \gamma)$ are neglected. Therefore,

$$\int_0^H P_r\ (r,\ \gamma)\ \phi_1\ (\gamma)\ d\gamma\ =\ P_r(r)\ \phi_1\ (0) \qquad (3.20a)$$

$$\int_0^H P_\gamma\ (r,\ \gamma)\,\psi_1\ (\gamma)\ d\gamma\ =\ P_\gamma(r)\,\psi_1\ (0) \qquad (3.20b)$$

3.4 SHELL FOUNDATION INTERACTION EQUATIONS:

3.4.1 <u>Thin Shallow Spherical Shell on Elastic Foundation</u>:

Consider a thin shallow spherical shell on an elastic foundation (Fig.3.2a) carrying axisymmetric loads. It is assumed that the friction between the shell and the foundation is negligibly small. Hence the shell transfers only normal reaction on to the supporting medium. Thus the differential Eqns. (3.6) for thin shallow spherical shell can be written as,

$$\frac{d^2u_s}{dr^2} + \frac{1}{r}\frac{du_s}{dr} - \frac{u_s}{r^2} + \frac{(1-\mu_s)}{R^2}u_s + \frac{(1+\mu_s)}{R}\frac{dw_s}{dR}$$

$$= -\frac{(1-\mu_s^2)}{E_s\,h}\,X \qquad (3.21a)$$

$$-\frac{h^2}{12}\left[\frac{d^4w_s}{dr^4} + \frac{2}{r}\frac{d^3w_s}{dr^3} - \frac{1}{r^2}\frac{d^2w_s}{dr^2} + \frac{2}{R^2}\frac{d^2w_s}{dr^2} + \frac{1}{r^3}\frac{dw_s}{dr}\right.$$

$$\left. + \frac{2}{rR^2}\frac{dw_s}{dr}\right] - \frac{2(1+\mu_s)}{R^2}w_s - \frac{(1+\mu_s)}{R}\frac{du_s}{dr}$$

$$-\frac{(1+\mu_s)}{Rr}u_s = \cdot\frac{(1-\mu_s^2)}{E_s\,h}\left[Z - P_\gamma(r)\right] \qquad (3.21b)$$

Where $P_\gamma(r)$ is the normal reaction due to elastic foundation on the shell.

As the shell is assumed to transfer only normal forces to the foundation, and assuming the compatibility of normal deflections of the shell and foundation surface, $P_\gamma(r)$ can be eliminated from Eqn. (3.21b) using Eqns. (3.20b), (3.18h) and (3.17b). Considering that $P_r(r)$ equals to zero in Eqn. (3.17a), the governing equations for the displacements of the shell foundation system can be written as [choosing $\phi_1(o) = \Psi_1(o) = 1$]

$$a_{11}\frac{d^2u_1}{dr^2} + \frac{a_{11}}{r}\frac{du_1}{dr} - \frac{a_{11}}{r^2}u_1 - b_{11}u_1 + d_{11}\frac{dw_1}{dr}$$

$$-c_{11}\frac{dw_1}{dr} = -\left[\sigma_{r\gamma}\;\phi_1'(\gamma)\right]_o^H \qquad (3.22a)$$

$$\frac{d^2u_s}{dr^2} + \frac{1}{r}\frac{du_s}{dr} - \frac{u_s}{r^2} + \frac{(1-\mu_s)\,u_s}{R^2} + \frac{(1+\mu_s)}{R}\frac{dw_1}{dr}$$

$$= - \frac{(1-\mu_s{}^2)}{E_s\,h}\,X \qquad (3.22b)$$

$$r_{11}\frac{d^2w_1}{dr^2} + \frac{r_{11}}{r}\frac{dw_1}{dr} - s_{11}\,w_1 - d_{11}\frac{du_1}{dr} + c_{11}\frac{du_1}{dr}$$

$$+ c_{11}\frac{u_1}{r} - d_{11}\frac{u_1}{r} - \frac{E_s\,h}{(1-\mu_s{}^2)}\,\frac{2(1+\mu_o)}{E_o}$$

$$\Big[\,\frac{h^2}{12}\Big\{\frac{d^4w_1}{dr^4} + \frac{2}{r}\frac{d^3w_1}{dr^3} - \frac{1}{r^2}\frac{d^2w_1}{dr^2} + \frac{2}{R^2}\frac{d^2w_1}{dr^2}$$

$$+ \frac{1}{r^3}\frac{dw_1}{dr} + \frac{2}{rR}\frac{dw_1}{dr}\Big\} + \frac{2(1+\mu_s)}{R^2}\,w_1 + \frac{(1+\mu_s)}{R}\frac{du_s}{dr}$$

$$+ \frac{(1+\mu_s)}{Rr}\,u_s\,\Big] = - \frac{2(1+\mu_o)}{E_o}\,\Big[\,Z + \Big\{\sigma_{\gamma\gamma}\,\psi_1(\gamma)\Big\}_o^H\,\Big]$$

where a_{11}, b_{11} etc. are given by Eqns. (3.18).

When free boundary condition for the shell is considered, the flank (portion of foundation beyond shell foundation) has to be taken care of [Fig. (3.2a)]. If u_f, w_f are the displacement functions for the flank portion, the governing differential equations for a free shallow spherical shell on an elastic foundation would be Eqns. (3.22) taken together with the following equations which accounts for the flank.

$$a_{11} \frac{d^2u_f}{dr^2} + \frac{a_{11}}{r} \frac{du_f}{dr} - \frac{a_{11}}{r^2} u_f \quad b_{11} u_f + d_{11} \frac{dw_f}{dr}$$

$$- c_{11} \frac{dw_f}{dr} = - [(\sigma_{r\gamma})_f \, \phi_1(\gamma)]_o^H \qquad (3.23a)$$

$$r_{11} \frac{d^2w_f}{dr^2} + \frac{r_{11}}{r} \frac{dw_f}{dr} - s_{11} w_f - d_{11} \frac{du_f}{dr} + c_{11} \frac{du_f}{dr}$$

$$+ c_{11} \frac{u_f}{r} - d_{11} \frac{u_f}{r} = -[(\sigma_{\gamma\gamma})_f \, \psi_1(r)]_o^H \qquad (3.23b)$$

where a_{11}, b_{11} etc. are given by Eqns. (3.18).

3.4.1.1 Choice of Dimensionless Displacement Functions:

In all the numerical examples that are considered, it is assumed that the elastic foundation consists of single elastic material and is of finite thickness H. The dimensionless functions $\phi_1(\gamma)$ and $\psi_1(\gamma)$ have to be chosen from the physical constraints of the problem. In the present investigation, for finite elastic layers, the following functions are used,

$$\phi_1(\gamma) = \psi_1(\gamma) = \frac{H-\gamma}{H} \qquad (3.24)$$

Substituting $\phi_1(\gamma)$ and $\psi_1(\gamma)$ from Eqn. (3.24) in Eqns.(3.18) for a_{11}, b_{11} etc. and integrating, the resulting expressions are,

$$a_{11} = \frac{E_o}{(1 - \mu_o^2)} \frac{H}{3} \qquad (3.25a)$$

$$b_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{1}{H} \tag{3.25b}$$

$$c_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{1}{2} \tag{3.25c}$$

$$d_{11} = \frac{E_o \mu_o}{(1 - \mu_o^2)} \frac{1}{2} \tag{3.25d}$$

$$r_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{H}{3} \tag{3.25e}$$

$$s_{11} = \frac{E_o}{(1 - \mu_o^2)} \frac{1}{H} \tag{3.25f}$$

In order to non-dimensionalise the Eqns. (3.22) and (3.23) the following non-dimensionalised functions are assumed,

$$\bar{u}_s = u_s/h \; ; \quad \bar{u}_1 = u_1/h, \quad \bar{w}_1 = w_1/h, \; \bar{u}_f = u_f/h,$$

$$\bar{w}_f = w_f/h, \quad \bar{r} = r/a \tag{3.26}$$

where a is the radius of the shell and h is the thickness of the shell.

Stresses $\sigma_{r\gamma}$, $\sigma_{\gamma\gamma}$ are zero at the surface of the foundation and flank and at $\gamma = H$, $\phi_1(\gamma) = \Psi_1(\gamma) = 0$ [Eqn. (3.24)]. From above it can be seen that,

$$[\sigma_{r\gamma}\, \phi_1(\gamma)]_o^H = [\sigma_{\gamma\gamma} \Psi_1(\gamma)]_o^H = [(\sigma_{r\gamma})_f\, \phi_1(\gamma)]_o^H$$

$$= [(\sigma_{\gamma\gamma})_f \Psi_1(\gamma)]_o^H = 0 \tag{3.27}$$

If the shell is subjected to only normal load transferred by the column, coming as ring load, then the components of the forces X, Z are all equal to zero.

Taking care of the facts stated above and substituting for a_{11}, b_{11} etc. from Eqns. (3.25) and also using the dimensionless functions given by Eqns. (3.26), the differential Eqns. (3.22) and (3.23) can be written in dimensionless form as given below.

Differential Eqns. (3.28a), (3.28b) are for the flank portion and are used only in case of free shell along with other equations,

$$\frac{d^2\bar{u}_f}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_f}{d\bar{r}} + \frac{\bar{u}_f}{\bar{r}^2} + \frac{3}{2}(1-\mu_o)\left(\frac{a}{H}\right)^2 u_f - \frac{3}{2}\mu_o\left(\frac{a}{H}\right)\frac{d\bar{w}_f}{d\bar{r}} + \frac{3}{4}(1-\mu_o)\left(\frac{a}{H}\right)\frac{d\bar{w}_f}{d\bar{r}} \tag{3.28a}$$

$$\frac{d^2\bar{w}_f}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{w}_f}{d\bar{r}} + \frac{6}{(1-\mu_o)}\left(\frac{a}{H}\right)^2\bar{w}_f + \frac{3\mu_o}{(1-\mu_o)}\left(\frac{a}{H}\right)\frac{d\bar{u}_f}{d\bar{r}} - \frac{3}{2}\left(\frac{a}{H}\right)\frac{d\bar{u}_f}{d\bar{r}} - \frac{3}{2}\left(\frac{a}{H}\right)\frac{\bar{u}_f}{\bar{r}} + \frac{3\mu_o}{(1-\mu_o)}\left(\frac{a}{H}\right)\frac{\bar{u}_f}{\bar{r}} \tag{3.28b}$$

$$\frac{d^2\bar{u}_1}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_1}{d\bar{r}} + \frac{\bar{u}_1}{\bar{r}^2} + \frac{3}{2}(1-\mu_o)\left(\frac{a}{H}\right)^2\bar{u}_1 - \frac{3}{2}\mu_o\left(\frac{a}{H}\right)\frac{d\bar{w}_1}{d\bar{r}} + \frac{3}{4}(1-\mu_o)\left(\frac{a}{H}\right)\frac{d\bar{w}_1}{d\bar{r}} \tag{3.28c}$$

$$\frac{d^2\bar{u}_s}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_s}{d\bar{r}} + \frac{\bar{u}_s}{\bar{r}^2} - (1 + \mu_s)\left(\frac{a}{R}\right)\frac{d\bar{w}_1}{d\bar{r}}$$

$$- (1 - \mu_s)\left(\frac{a}{R}\right)^2 \bar{u}_s \qquad (3.28d)$$

$$\frac{d^4\bar{w}_1}{d\bar{r}^4} = \left[\frac{E_o'}{E_s}\frac{6(1-\mu_s^2)}{(1+\mu_o)}\right]\left[\left(\frac{R}{h}\right)^3\left(\frac{a}{R}\right)^3\right]\left[\frac{1}{3}\left(\frac{H}{a}\right)\frac{d^2\bar{w}_1}{d\bar{r}^2}\right.$$

$$+ \frac{1}{3}\left(\frac{H}{a}\right)\frac{1}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}} - \frac{2}{(1-\mu_o)}\left(\frac{a}{H}\right)\bar{w}_1$$

$$\left. - \frac{\mu_o}{(1-\mu_o)}\frac{d\bar{u}_1}{d\bar{r}} + \frac{1}{2}\frac{du_1}{d\bar{r}} + \frac{1}{2}\frac{\bar{u}_1}{\bar{r}} - \frac{\mu_o}{(1-\mu_o)}\frac{\bar{u}_1}{\bar{r}}\right]$$

$$- \frac{2}{\bar{r}}\frac{d^3\bar{w}_1}{d\bar{r}^3} + \frac{1}{\bar{r}^2}\frac{d^2\bar{w}_1}{d\bar{r}^2} - 2\left(\frac{a}{R}\right)^2\frac{d^2\bar{w}_1}{d\bar{r}^2} -$$

$$- 2\left(\frac{a}{R}\right)^2\frac{1}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}} - \frac{1}{\bar{r}^3}\frac{d\bar{w}_1}{d\bar{r}} - 24\,(1 + \mu_s)\left(\frac{R}{h}\right)^2$$

$$\times \left(\frac{a}{R}\right)^4 \bar{w}_1 - 12\,(1 + \mu_s)\left(\frac{R}{h}\right)^2\left(\frac{a}{R}\right)^3\frac{d\bar{u}_s}{d\bar{r}}$$

$$- 12\,(1 + \mu_s)\left(\frac{R}{h}\right)^2\left(\frac{a}{R}\right)^3\frac{\bar{u}_s}{\bar{r}} \qquad (3.28e)$$

3.4.1.2 Forces and Moments in Shell:

Forces and moments in dimensionless form in spherical shells for axisymmetric loading can be written as,

$$\bar{N}_1 = \frac{N_1}{E_s h} = \frac{1}{(1-\mu_s^2)}\left(\frac{h}{R}\right)\left[\left(\frac{R}{a}\right)\frac{d\bar{u}_s}{d\bar{r}} + \bar{w}_1 + \mu_s\left(\frac{R}{a}\right)\frac{\bar{u}_s}{\bar{r}}\right.$$

$$\left. + \mu_s\,\bar{w}_1\right] \qquad (3.29a)$$

$$\bar{N}_2 = \frac{N_2}{E_s h} = \frac{1}{(1-\mu_s^2)} \left(\frac{h}{R}\right) \left[\left(\frac{R}{a}\right) \frac{\bar{u}_s}{\bar{r}} + \bar{w}_1 + \mu_s \left(\frac{R}{a}\right) \frac{d\bar{u}_s}{d\bar{r}} + \mu_s \bar{w}_1 \right] \qquad (3.29b)$$

$$\bar{M}_1 = \frac{M_1 a}{D} = \left(\frac{h}{R}\right)\left(\frac{R}{a}\right) \left[\frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{\mu_s}{\bar{r}} \frac{d\bar{w}_1}{d\bar{r}} \right] \qquad (3.29c)$$

$$\bar{M}_2 = \frac{M_2 a}{D} = \left(\frac{h}{R}\right)\left(\frac{R}{a}\right) \left[\frac{1}{\bar{r}} \frac{d\bar{w}_1}{d\bar{r}} + \mu_s \frac{d^2\bar{w}_1}{d\bar{r}^2} \right] \qquad (3.29d)$$

$$\bar{q} = \frac{2(1+\mu_o)P_\gamma(r)}{E_o} = -\left(\frac{h}{R}\right)\left(\frac{R}{a}\right) \left[-\frac{1}{3} \left(\frac{H}{a}\right) \frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{1}{3}\left(\frac{H}{a}\right) \frac{1}{\bar{r}} \frac{d\bar{w}_1}{d\bar{r}} - \frac{2}{(1-\mu_o)} \left(\frac{a}{H}\right) \bar{w}_1 - \frac{\mu_o}{(1-\mu_o)} \frac{d\bar{u}_1}{d\bar{r}} + \frac{1}{2} \frac{d\bar{u}_1}{d\bar{r}} + \frac{1}{2} \frac{\bar{u}_1}{\bar{r}} - \frac{\mu_o}{(1-\mu_o)} \frac{\bar{u}_1}{\bar{r}} \right] \qquad (3.29e)$$

where,

$$D = \frac{E_s h^3}{12(1-\mu_s^2)} \qquad (3.29f)$$

3.4.2 Thin Shallow Conical Shells on Elastic Foundations:

Proceeding on the similar lines, as in case of spherical shells, the governing differential equations in dimensionless form for the conical shell on elastic foundation can be written as given below [See Fig. (3.2b)].

Differential equations for flank portion are:

$$\frac{d^2\bar{u}_f}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_f}{d\bar{r}} + \frac{\bar{u}_f}{\bar{r}^2} + \frac{3}{2}(1-\mu_o)\left(\frac{a}{H}\right)^2\bar{u}_f - \frac{3}{2}\mu_o\left(\frac{a}{H}\right)\frac{d\bar{w}_f}{d\bar{r}}$$

$$+\frac{3}{4}(1-\mu_o)\left(\frac{a}{H}\right)\frac{d\bar{w}_f}{d\bar{r}} \qquad (3.30a)$$

$$\frac{d^2\bar{w}_1}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{dw_f}{d\bar{r}} + \frac{6}{(1-\mu_o)}\left(\frac{a}{H}\right)^2\bar{w}_f + \frac{3\mu_o}{(1-\mu_o)}\left(\frac{a}{H}\right)\frac{d\bar{u}_f}{d\bar{r}}$$

$$-\frac{3}{2}\left(\frac{a}{H}\right)\frac{d\bar{u}_f}{d\bar{r}} - \frac{3}{2}\left(\frac{a}{H}\right)\frac{\bar{u}_f}{\bar{r}} + \frac{3\mu_o}{(1-\mu_o)}\left(\frac{a}{H}\right)\frac{\bar{u}_f}{\bar{r}}$$

$$(3.30b)$$

Differential equations for shell-foundation system are

$$\frac{d^2\bar{u}_1}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_1}{d\bar{r}} + \frac{\bar{u}_1}{\bar{r}^2} + \frac{3}{2}(1-\mu_o)\left(\frac{a}{H}\right)^2\bar{u}_1$$

$$-\frac{3}{2}\mu_o\left(\frac{a}{H}\right)\frac{d\bar{w}_1}{d\bar{r}} + \frac{3}{4}(1-\mu_o)\left(\frac{a}{H}\right)\frac{d\bar{w}_1}{d\bar{r}} \qquad (3.31a)$$

$$\frac{d^2\bar{u}_s}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_s}{d\bar{r}} + \frac{\bar{u}_s}{\bar{r}^2} - \mu_s\frac{\cos\theta}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}} + \frac{\cos\theta}{\bar{r}^2}\bar{w}_1 \qquad (3.31b)$$

$$\frac{d^4\bar{w}_1}{d\bar{r}^4} = \frac{E_o}{E_s}\frac{6(1-\mu_s^2)}{(1+\mu_o)}\left(\frac{a}{h}\right)^3\left[\frac{1}{3}\left(\frac{H}{a}\right)\frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{1}{3}\left(\frac{H}{a}\right)\frac{1}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}}\right.$$

$$\left.-\frac{2}{(1-\mu_o)}\left(\frac{a}{H}\right)\bar{w}_1 - \frac{\mu_o}{(1-\mu_o)}\frac{d\bar{u}_1}{d\bar{r}} + \frac{1}{2}\frac{d\bar{u}_1}{d\bar{r}}\right.$$

$$+ \frac{1}{2} \frac{\bar{u}_1}{\bar{r}} - \frac{\mu_o}{(1 - \mu_o)} \frac{\bar{u}_1}{\bar{r}}] \cdot \frac{2}{\bar{r}} \frac{d^3\bar{w}_1}{d\bar{r}^3} + \frac{(1 - Cos^2\theta)}{\bar{r}^2} \frac{d^2\bar{w}_1}{d\bar{r}^2}$$

$$- \frac{(1 + Cos^2 \theta)}{\bar{r}^3} \frac{d\bar{w}_1}{d\bar{r}} - \frac{4 \ Cos^2 \theta}{\bar{r}^4} \bar{w}_1 - 12 \left(\frac{a}{h}\right)^2 \frac{Cos^2 \theta}{\bar{r}^2} \bar{w}_1$$

$$- \frac{12}{\bar{r}} \mu_s \left(\frac{a}{h}\right)^2 Cos \ \theta \ \frac{d\bar{u}_s}{d\bar{r}} - \frac{12}{\bar{r}^2} \left(\frac{a}{h}\right)^2 Cos \ \theta \ \bar{u}_s \qquad (3.31c)$$

where $\bar{u}_f$, $\bar{w}_f$ etc. are defined by Eqn. (3.26).

Differential Equations (3.31) are solved for fixed and simply supported boundary conditions. Differential equation (3.31) and (3.30) together are solved for free boundary condition.

Forces and moments in dimensionless form in conical shells for axisymmetric loading can be written as

$$\bar{N}_1 = \frac{N_1}{E_s h} = \frac{1}{(1 - \mu_s^2)} \left(\frac{h}{a}\right) \left[\frac{d\bar{u}_s}{d\bar{r}} + \mu_s \frac{\bar{u}_s}{\bar{r}} + \mu_s \frac{\bar{w}_1}{\bar{r}} Cos \ \theta\right] \qquad (3.32a)$$

$$\bar{N}_2 = \frac{N_2}{E_s h} = \frac{1}{(1 - \mu_s^2)} \left(\frac{h}{a}\right) \left[\frac{\bar{u}_s}{\bar{r}} + \frac{\bar{w}_1}{\bar{r}} Cos \ \theta + \mu_s \frac{d\bar{u}_s}{d\bar{r}}\right] \qquad (3.32b)$$

$$\bar{M}_1 = \frac{M_1 \ a}{D} = \left(\frac{h}{a}\right) \left[\frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{\mu_s}{\bar{r}} \frac{d\bar{w}_1}{d\bar{r}}\right] \qquad (3.32c)$$

$$\bar{M}_2 = \frac{M_2 \ a}{D} = \left(\frac{h}{a}\right) \left[\frac{1}{\bar{r}} \frac{d\bar{w}_1}{d\bar{r}} + \mu_s \frac{d^2\bar{w}_1}{d\bar{r}^2}\right] \qquad (3.32d)$$

$$\bar{q} = \frac{2(1+\mu_o)\, P_\gamma(r)}{E_o} = -\left(\frac{h}{a}\right)\Big[\frac{1}{3}\left(\frac{H}{a}\right)\frac{d^2\bar{w}_1}{d\bar{r}^2}$$

$$+ \frac{1}{3}\left(\frac{H}{a}\right)\frac{1}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}} - \frac{2}{(1-\mu_o)}\left(\frac{a}{H}\right)\bar{w}_1 - \frac{\mu_o}{(1-\mu_o)}\frac{d\bar{u}_1}{d\bar{r}}$$

$$+ \frac{1}{2}\frac{d\bar{u}_1}{d\bar{r}} + \frac{1}{2}\frac{\bar{u}_1}{d\bar{r}} - \frac{\mu_o}{(1-\mu_o)}\frac{\bar{u}_1}{\bar{r}}\Big] \qquad (3.32e)$$

where,

$$D = \frac{E_s h^3}{12\,(1-\mu_s^{\,2})} \qquad (3.32f)$$

3.5 BOUNDARY CONDITIONS:

Three important types of boundary conditions namely, simply supported (S.S.), fixed and free, have been considered for both conical and spherical shell.

3.5.1 Boundary Conditions at the Outer Boundary of Shell:

3.5.1.1 Simply Supported: Following are the boundary conditions.

(a) $u_1 = 0$ (3.33a)

(b) $u_s = 0$ (3.33b)

(c) $w_1 = 0$ (3.33c)

(d) $M_1 = 0$ (3.33d)

In dimensionless form, the above mentioned boundary conditions for both spherical and conical shells can be written as,

(a) $\bar{u}_1 = 0$ (3.34a)

(b) $\bar{u}_s = 0$ (3.34b)

(c) $\bar{w}_1 = 0$ (3.34c)

(d) $\dfrac{d^2\bar{w}_1}{d\bar{r}^2} + \dfrac{\mu_s}{\bar{r}} \dfrac{d\bar{w}_1}{d\bar{r}} = 0$ (3.34d)

3.5.1.2 Fixed or Built-in:

Following would be the boundary conditions for fixed case,

(a) $u_1 = 0$ (3.35a)

(b) $u_s = 0$ (3.35b)

(c) $w_1 = 0$ (3.35c)

(d) $\dfrac{dw_1}{dr} = 0$ (3.35d)

In dimensionless form, the above mentioned boundary conditions for both conical and spherical shells can be written as,

(a) $\bar{u}_1 = 0$ (3.36a)

(b) $\bar{u}_s = 0$ (3.36b)

(c) $\bar{w}_1 = 0$ (3.36c)

(d) $\dfrac{d\bar{w}_1}{d\bar{r}} = 0$ (3.36d)

3.5.1.3 Free:

(1) In free type of boundary conditions the flank portion of the foundation needs to be considered. The boundary conditions at the far end of flank can be written as

(a) $w_f = 0$ (3.37a)

(b) $u_f = 0$ (3.37b)

(2) The boundary conditions at the free end of shell can be written as,

(a) $u_f = u_1$ (3.38a)

(b) $w_f = w_1$ (3.38b)

(c) Transverse shear in flank

= Transverse shear in foundation

+ Transverse shear in shell (3.38c)

(d) Normal force in flank

= Normal force in foundation (3.38d)

(e) Normal force in shell = 0 (3.38e)

(f) $M_1 = 0$ (3.38f)

Above mentioned boundary conditions in dimensionless form for spherical shell can be written as,

(1) at the far end of the flank

(a) $\bar{u}_f = 0$ (3.39a)

(b) $\bar{w}_f = 0$ (3.39b)

(2) at the free end of the shell

(a) $\bar{u}_f = \bar{u}_1$ (3.40a)

(b) $\bar{w}_f = \bar{w}_1$ (3.40b)

(c)
$$\frac{E_o}{E_s} \frac{6(1-\mu_s^2)}{(1+\mu_o)} \left[\frac{1}{2} \bar{u}_f - \frac{1}{2} \bar{u}_1 + \frac{1}{3} \left(\frac{H}{a}\right) \frac{d\bar{w}_f}{d\bar{r}} - \frac{1}{3} \left(\frac{H}{a}\right) \frac{d\bar{w}_1}{d\bar{r}} \right] + \left(\frac{h}{R}\right)^3 \left(\frac{R}{a}\right)^3 \left[\frac{d^3\bar{w}_1}{d\bar{r}^3} + \frac{1}{\bar{r}} \frac{d^2\bar{w}_1}{d\bar{r}^2} - \frac{1}{\bar{r}^2} \frac{d\bar{w}_1}{d\bar{r}} \right] = 0 \quad (3.40c)$$

(d)
$$\frac{d\bar{u}_f}{d\bar{r}} - \frac{d\bar{u}_1}{d\bar{r}} + \mu_o \left[\frac{\bar{u}_f}{\bar{r}} - \frac{\bar{u}_1}{\bar{r}} + \frac{3}{2} \left(\frac{a}{H}\right) \bar{w}_f \quad \frac{3}{2} \left(\frac{a}{H}\right) \bar{w}_1 \right] \quad (3.40d)$$

(e)
$$\frac{d\bar{u}_s}{d\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_1 + \mu_s \left[\frac{\bar{u}_s}{\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_1 \right] = 0 \quad (3.40e)$$

(f)
$$\frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{\mu_s}{\bar{r}} \frac{d\bar{w}_1}{d\bar{r}} = 0 \quad (3.40f)$$

Boundary conditions in free case for conical shell in dimensionless form can be written as

(1) at the far end of the flank

(a) $\bar{u}_f = 0$ (3.41a)

(b) $\bar{w}_f = 0$ (3.41b)

(2) at the free end of the shell

(a) $\bar{u}_f = \bar{u}_1$ (3.42a)

(b) $\bar{w}_f = \bar{w}_1$ (3.42b)

(c) $$\frac{E_o}{E} \frac{6(1-\mu^2)}{(1+\mu_o)} \left(\frac{a}{h}\right)^3 \left[\frac{1}{2}\bar{u}_f - \frac{1}{2}\bar{u}_1 + \frac{1}{3}\left(\frac{H}{a}\right)\frac{d\bar{w}_f}{d\bar{r}} - \frac{1}{3}\left(\frac{H}{a}\right)\frac{d\bar{w}_1}{d\bar{r}} \right] - \frac{d^3\bar{w}_1}{d\bar{r}^3} - \frac{1}{\bar{r}} \cdot \frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{1}{\bar{r}^2}\frac{d\bar{w}_1}{d\bar{r}} = 0 \quad (3.42c)$$

(d) $$\frac{d\bar{u}_1}{d\bar{r}} - \frac{d\bar{u}_f}{d\bar{r}} + \mu_o \frac{\bar{u}_1}{\bar{r}} - \mu_o \frac{\bar{u}_f}{\bar{r}} + \frac{3}{2}\mu_o \left(\frac{a}{H}\right)\bar{w}_1 - \frac{3}{2}\mu_o\left(\frac{a}{H}\right)\bar{w}_f = 0 \quad (3.42d)$$

(e) $$\frac{d\bar{u}_s}{d\bar{r}} + \mu_s \left[\frac{\bar{u}_s}{\bar{r}} + \frac{\bar{w}_1 \operatorname{Cos} \theta}{\bar{r}} \right] = 0 \quad (3.42e)$$

(f) $$\frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{\mu_s}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}} = 0 \quad (3.42f)$$

3.5.2 Boundary Conditions at the Column End:

Boundary conditions at the column end of the shell can be written as,

(a) Transverse shear in shell

+ Transverse shear in foundation

= Load/unit length (3.43a)

(b) $u_1 = 0$ (3.43b)

(c) $u_s = 0$ (3.43c)

(d) $\frac{dw_1}{dr} = 0$ (3.43d)

In non-dimensional form, the above mentioned boundary conditions can be written as,

(1) Spherical shell

(a) $$\frac{E_o}{E_s}\ \frac{6(1-\mu_s^{\,2})}{(1-\mu_o)}\ (\frac{R}{h})(\frac{a}{R})\ [\frac{1}{2}\ \bar{u}_1 + \frac{1}{3}\ (\frac{H}{a})\ \frac{d\bar{w}_1}{d\bar{r}}]$$

$$-\ (\frac{h}{R})^2\ (\frac{R}{a})^2\ [\ \frac{d^3\bar{w}_1}{d\bar{r}^3} + \frac{1}{\bar{r}}\ \frac{d^2\bar{w}_1}{d\bar{r}^2} - \frac{1}{\bar{r}^2}\ \frac{d\bar{w}_1}{d\bar{r}}\ ]$$

$$= \frac{6(1-\mu_s^{\,2})}{\pi}\ [\,(\frac{a}{h})(-\frac{P}{E_s h^2})\,] \qquad (3.44a)$$

where P is the load transferred by the column on shell foundation,

(b) $\bar{u}_1 = 0$ (3.44b)

(c) $\bar{u}_s = 0$ (3.44c)

(d) $\frac{d\bar{w}_1}{d\bar{r}} = 0$ (3.44d)

(2) Conical shell

(a) $$\frac{E_o}{E_s}\ \frac{6(1-\mu_s^{\,2})}{(1-\mu_o)}\ (\frac{a}{h})^3\ [\ \frac{1}{2}\ \bar{u}_1 + \frac{1}{3}\ (\frac{H}{a})\ \frac{d\bar{w}_1}{d\bar{r}}\ ]$$

$$[\ \frac{d^3\bar{w}_1}{d\bar{r}^3} + \frac{1}{\bar{r}}\ \frac{d^2\bar{w}_1}{d\bar{r}^2} - \frac{1}{\bar{r}^2}\ \frac{d\bar{w}_1}{d\bar{r}}\ ]$$

$$= \frac{6(1-\mu_s^{\,2})}{\pi}\ [(\frac{a}{b})(-\frac{P}{E_s h^2})] \qquad (3.45a)$$

(b) $\bar{u}_1 = 0$ (3.45b)

(c) $\bar{u}_s = 0$ (3.45c)

(d) $\frac{d\bar{w}_1}{d\bar{r}} = 0$ (3.45d)

In both spherical shell and conical shell, $\left[\left(\frac{a}{b}\right)\left(\frac{P}{E_s h^2}\right)\right]$ has been taken as the load parameter.

3.6 NUMERICAL CALCULATIONS AND CONCLUSIONS:

In all the numerical examples that have been solved, following parameter values have been kept constant:

$$\frac{H}{a} = 2.0$$

$$\frac{b}{a} = 0.2$$

$$\mu_s = 0.25$$

$$\mu_o = 0.17$$

In case of spherical shell, $\left(\frac{R}{a}\right) = 4.0$ has been kept constant while $\left(\frac{h}{R}\right)$ and $\left(\frac{E_s}{E_o}\right)$ have been varied. Also $\left(\frac{h}{R}\right) = 0.0333$ has been kept constant while $\left(\frac{R}{a}\right)$ and $\left(\frac{E_s}{E_o}\right)$ have been varied.

Similarly in case of conical shells $\left(\frac{a}{h}\right) = 14.0$ is a constant while θ and $\frac{E_s}{E_o}$ take different values.

Results have been presented in plots drawn between $\bar{r}$ and functional values at unit load parameter. Load parameter has been taken as,

$$\widetilde{P} = \left(\frac{a}{b}\right)\left(\frac{P}{E_s h^2}\right) \tag{3.46}$$

Functions have been redefined as follows:

$$\widetilde{u}_1 = \bar{u}_1/\widetilde{P} \tag{3.47a}$$

$$\widetilde{u}_s = \bar{u}_s/\widetilde{P} \tag{3.47b}$$

$$\widetilde{w}_1 = \bar{w}_1/\widetilde{P} \tag{3.47c}$$

$$\widetilde{N}_1 = \bar{N}_1/\widetilde{P} \tag{3.47d}$$

$$\widetilde{N}_2 = \bar{N}_2/\widetilde{P} \tag{3.47e}$$

$$\widetilde{M}_1 = \bar{M}_1/\widetilde{P} \tag{3.47f}$$

$$\widetilde{M}_2 = \bar{M}_2/\widetilde{P} \tag{3.47g}$$

$$\widetilde{q} = \bar{q}/\widetilde{P} \tag{3.47h}$$

Dimensionless displacements and forces $\widetilde{u}_1$, $\widetilde{u}_s$, $\widetilde{w}_1$, $\widetilde{N}_1$, $\widetilde{N}_2$, $\widetilde{M}_1$, $\widetilde{M}_2$ and $\widetilde{q}$ for shallow spherical and conical shells on elastic foundations have been presented for simply supported, fixed and free types of boundary conditions.

3.6.1 Simply Supported and Fixed Boundary Conditions:

(a) Spherical Shell:

From Figs. (3.3a), (3.3b), (3.3c), (3.4a), (3.4b) and (3.4c) it can be seen that $\widetilde{u}_1$, $\widetilde{u}_s$ and $\widetilde{w}_1$ increases with

decrease in $(\frac{h}{R})$, also the values of functions decrease with decrease in $(\frac{E_g}{E_0})$ i.e. when foundation is stiffer. One important observation that can be made from Figs. (3.3a), (3.3b), (3.4a) and (3.4b) is that the radial displacements in shell are much larger than those in foundation. This happens because the shell is free to slip over the foundation as the friction between them is negligible. It can also be observed that the maximum values of $\tilde{u}_1$ and $\tilde{u}_s$ occur at same $\bar{P}$ for a particular parameter. It can also be observed that the values of $\tilde{u}_1$, $\tilde{u}_s$ and $\tilde{w}_1$ for fixed boundary conditions are less than those for simply supported boundary conditions.

Responses $\tilde{N}_1$, $\tilde{N}_2$ for simply supported spherical shell are shown in Figs. (3.3d) and (3.3e) where it can be seen that $\tilde{N}_1$ is negative and $\tilde{N}_2$ is positive for all values of $\bar{P}$ and they increase with increase of $(\frac{h}{R})$ value. Figs. (3.3f), (3.3g), (3.3h), (3.4d), (3.4e) and (3.4f) show the variation of $\tilde{M}_1$, $\tilde{M}_2$ and $\tilde{q}$ along $\bar{P}$. It can be observed that their values increase with increase in $(\frac{h}{R})$ ratio. It can be noted that variation of $\tilde{q}$ is not constant or near that. $\tilde{q}$ has maximum value near the column end and decreases away from it.

The responses $\tilde{u}_1$, $\tilde{u}_s$ increase with decrease in $(\frac{R}{a})$ as shown in Figs. (3.5a), (3.5b) where as $\tilde{w}_1$ increases with increase in $(\frac{R}{a})$ [Fig. (3.5c)]. $\tilde{u}_s$ is greater than $\tilde{u}_1$ because of the slip between shell and foundation. $\tilde{N}_1$ increases as $(\frac{R}{a})$

increases [Fig. (3.5d)] but reverse is the case for $\tilde{N}_2$ [Fig. (3.5e)]. As the shell goes deep (as $(\frac{R}{a})$ decreases), the stretching of the shell is more, and since the boundaries are not allowed to move, more negative $\tilde{M}_1$, $\tilde{M}_2$ and positive $\tilde{q}$ develops near the shell boundary [Figs.(3.5f),(3.5g),(3.5h),(3.6a),(3.6b), (3.6c)]. Also the absolute values of $\tilde{M}_1$, $\tilde{M}_2$ and $\tilde{q}$ decrease with increase in $(\frac{R}{a})$, except near the column end where values for all $(\frac{R}{a})$ are almost same. It can be observed that the values of $\tilde{M}_1$, $\tilde{M}_2$ and $\tilde{q}$ are greater for a greater $(\frac{E_s}{E_o})$ ratio.

(b) Conical Shells:

In case of simply supported and fixed conical shells, $\tilde{w}_1$ decreases with decrease in the value of θ, which means that normal deflections are decreasing as the shell goes deeper [Fig. (3.7a) and (3.8a)]. When $\theta = \frac{\pi}{2} \simeq 1.57$, the conical shell becomes a flat circular plate and $\tilde{w}$ at $\theta = 1.57$ is as large as 10 times of that at $\theta = 1.31$, as shown in Fig. (3.7a). It can also be observed that $\tilde{w}$ is higher for higher $\frac{E_s}{E_o}$ ratio which means that $\tilde{w}$ is having greater value when the foundation is less stiff.

Responses $\tilde{M}_1$, $\tilde{M}_2$ and q are having much greater value for plate than in shells and they decrease with decrease in θ value, as can be seen from Figs. (3.7b), (3.7c), (3.7d), (3.8b), (3.8c) and (3.8d).

3.6.2 Free Boundary Conditions:

In case of free boundary conditions, the flank (portion of foundation beyond the shell) has been considered. Results have been presented for flank widths equal to 1.6 times and 2.4 times the outer radius of the shell. From Figs.(3.9), (3.10) and (3.11) it can be observed that the responses coincide for the two flank widths considered. From this it can be concluded that the flank width equal to 1.6 times the outer radius of the shell is sufficient for solving any free boundary condition problem.

(a) Spherical Shell:

From Figs. (3.9a) and (3.9b) it can be seen that $\tilde{u}_s$ and $\tilde{w}_1$ increase with the increase in $(\frac{h}{R})$ ratio. In Fig.(3.9b) the dotted lines and firm lines represent normal displacement when width of the flank has been considered as 2.4 times and 1.6 times the outer radius of the shell respectively. It can be observed that the normal displacements in shell portion have the same value for both the flank widths considered. A kink in the curves at the boundary of flank and shell can be observed which is due to the change in slope. $\tilde{N}_1$ is slightly negative near the free boundary but becomes positive as it approaches the middle of the shell and its value increases with increase in $(\frac{h}{R})$ [Fig. (3.9c)]. Similar is the case for $\tilde{N}_2$ except that it is slightly positive

near the free boundary and becomes negative as it approaches column end [Fig. (3.9d)]. Figs. (3.9e) and (3.9f) show $\tilde{M}_1$, $\tilde{M}_2$ for different values of $(\frac{h}{R})$, and they are maximum at the least value of $(\frac{h}{R})$. From Fig. (3.9g), the maximum value of $\tilde{q}$ is same for all the values of $(\frac{h}{R})$ but the curve flattens as $(\frac{h}{R})$ increases. From Fig. (3.10a), it can be seen that $\tilde{w}_1$ increases with increase in $(\frac{R}{a})$. From Fig. (3.10b) it can be seen that the maximum value of $\tilde{M}_1$ are almost same for all the values of $(\frac{R}{a})$ that are considered, but $\tilde{M}_1$ flattens for higher values of $(\frac{R}{a})$. $\hat{q}$ increases with increase in $(\frac{R}{a})$ and the curve flattens for higher values of $(\frac{R}{a})$ [Fig.(3.10c)]. A general observation, in free case, can be made is that the values of all the responses are same (or very near that) for both the flank widths considered.

(b) Conical Shells:

In case of free conical shells, response $\tilde{w}_1$ increases with increase in θ [Fig. (3.11a)]. At $\theta = \frac{\pi}{2} \simeq 1.57$ maximum $\tilde{w}$ is equal to 6.65 [not given in figure where as at $\theta = 1.31$ it is equal to 0.77 only. Responses $\tilde{M}_1$, $\tilde{M}_2$ and $\tilde{q}$ as shown in Figs. (3.11b), (3.11c) and (3.11d) increase with increase in θ and are maximum for $\theta = \frac{\pi}{2}$. It can also be seen that the curves coincide for the two flank widths considered.

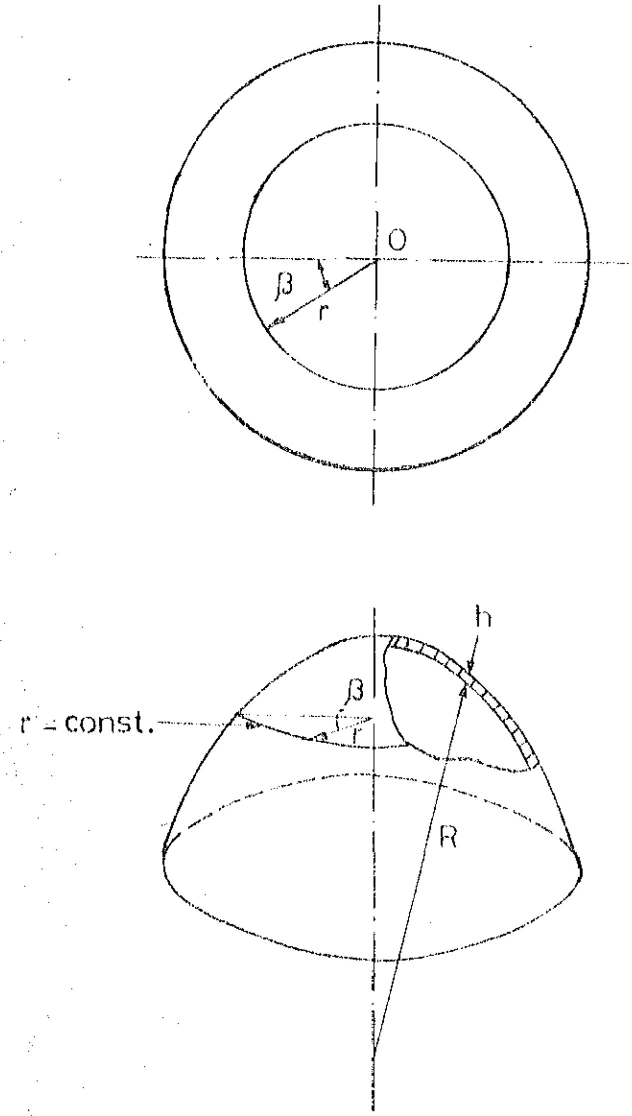


FIG. 3.1a SHALLOW SPHERICAL SHELL

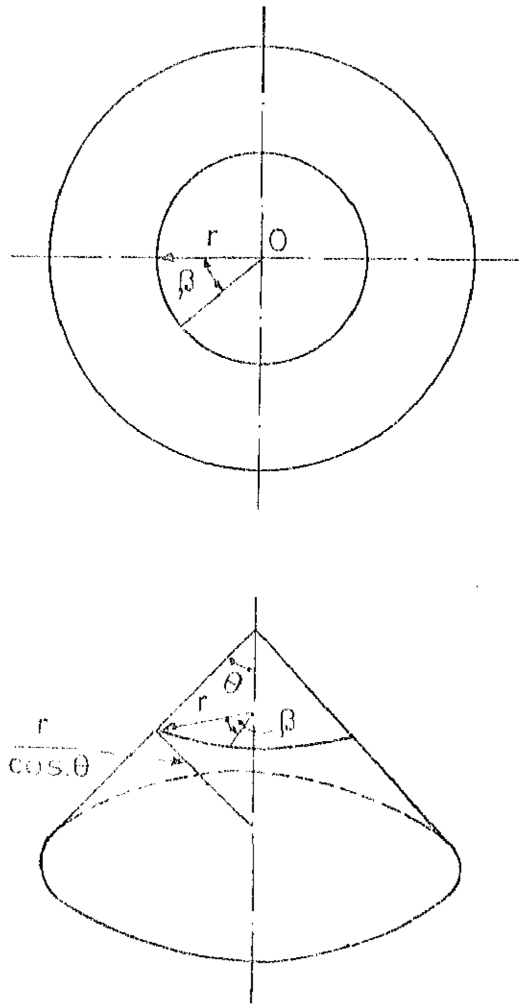


FIG. 3.1b SHALLOW CONICAL SHELL

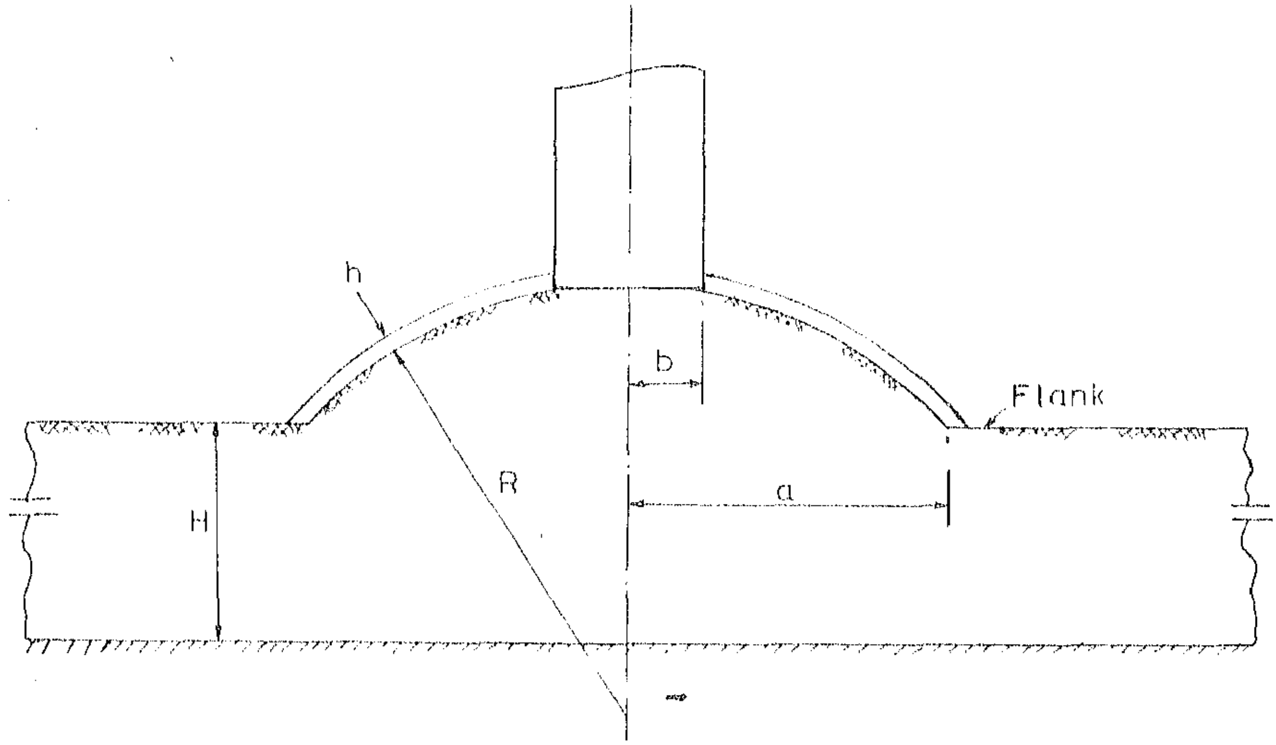


FIG. 3.2a SPHERICAL SHELL ON ELASTIC FOUNDATION

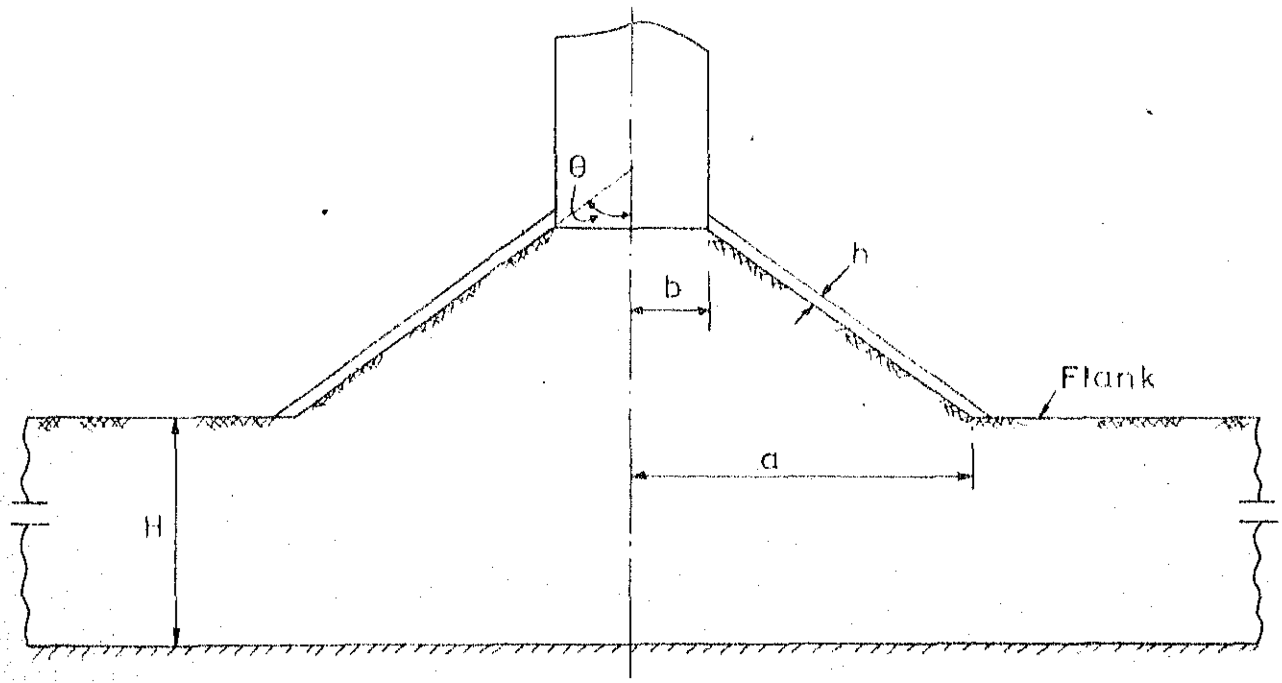


FIG. 3.2b CONICAL SHELL ON ELASTIC FOUNDATION

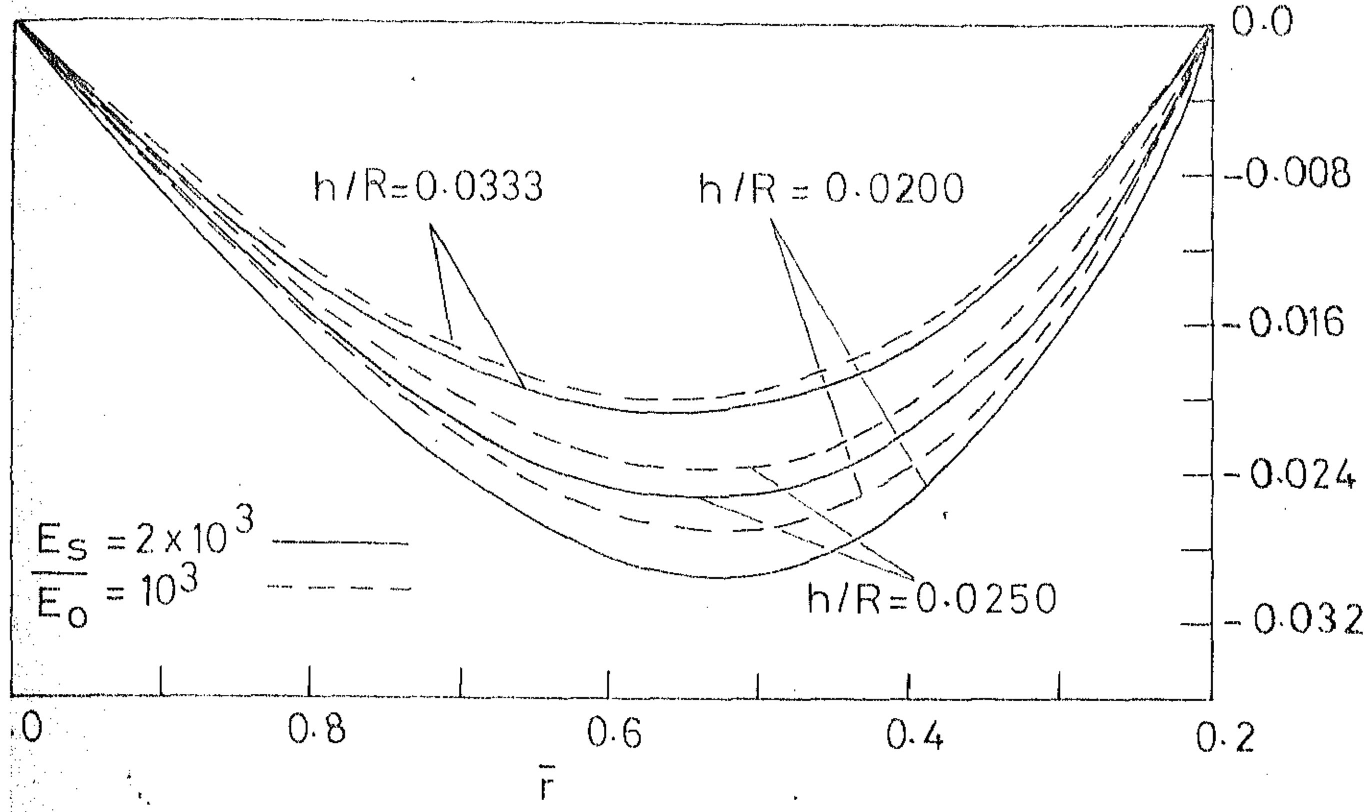


FIG. 3.3a SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{u}_1$ FOR DIFFERENT(h/R) AND (E_s/E_0)VALUES

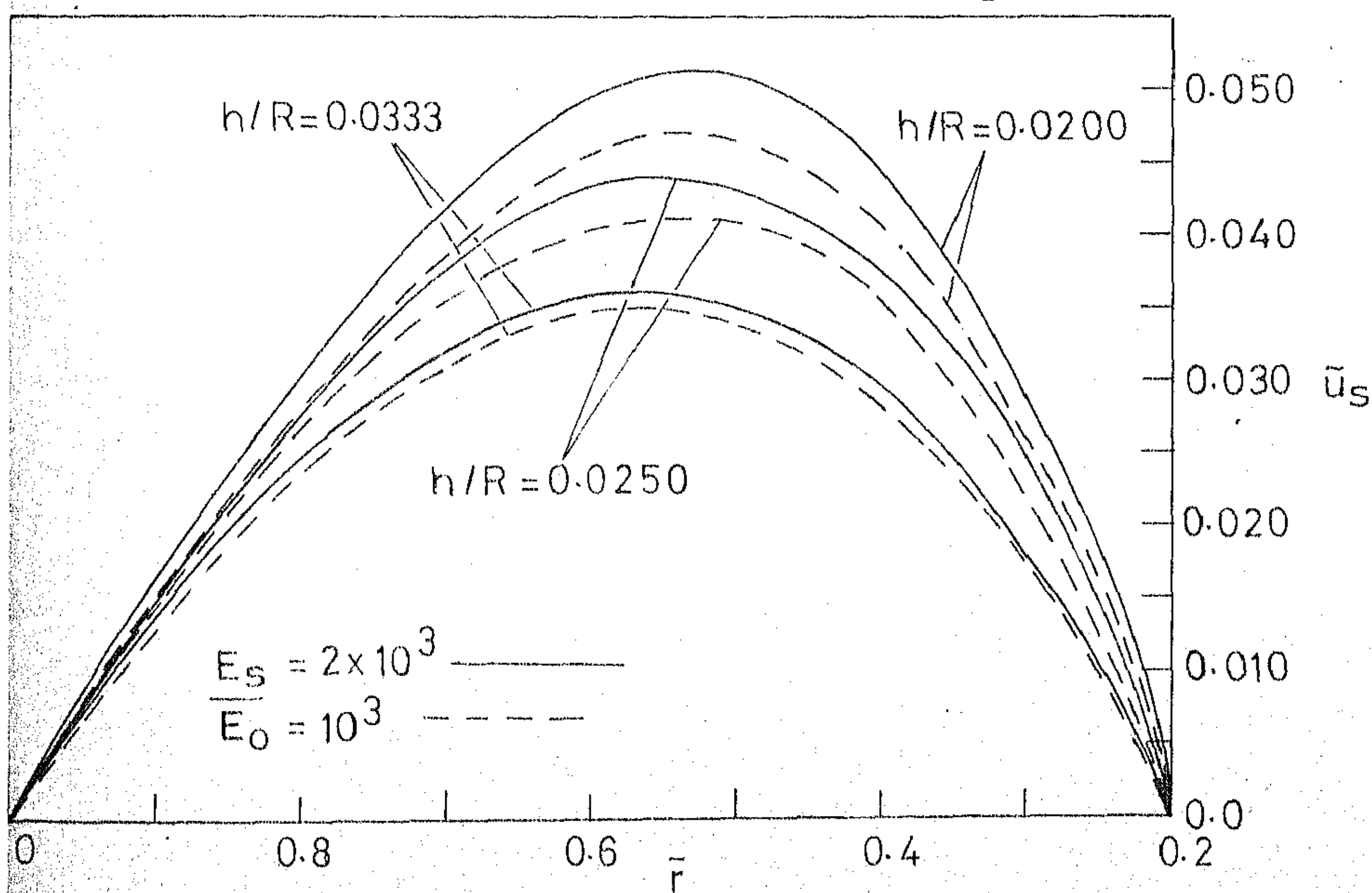


FIG. 3.3b SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{u}_s$ FOR DIFFERENT (h/R) AND(E_s/E_0)VALUES

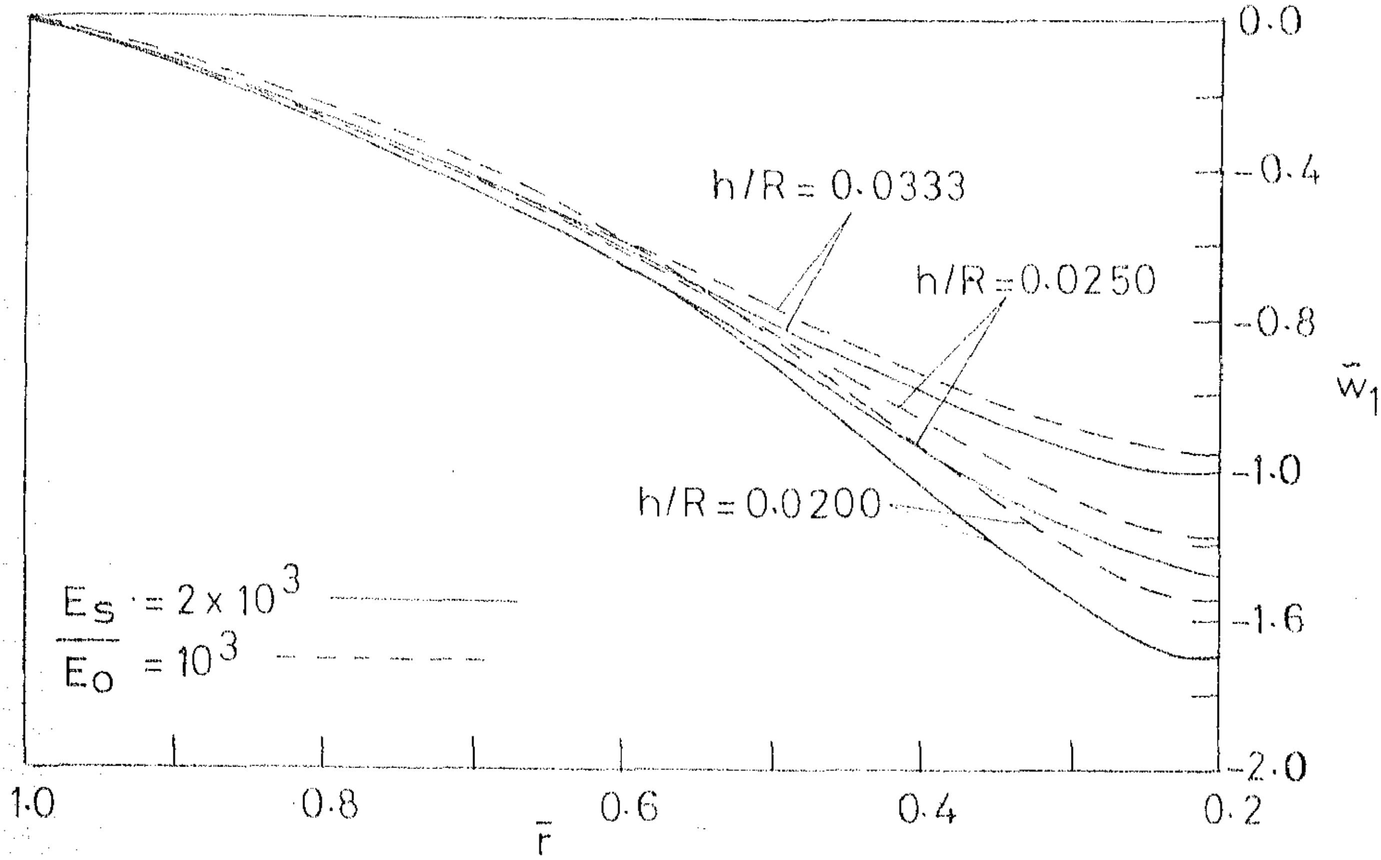
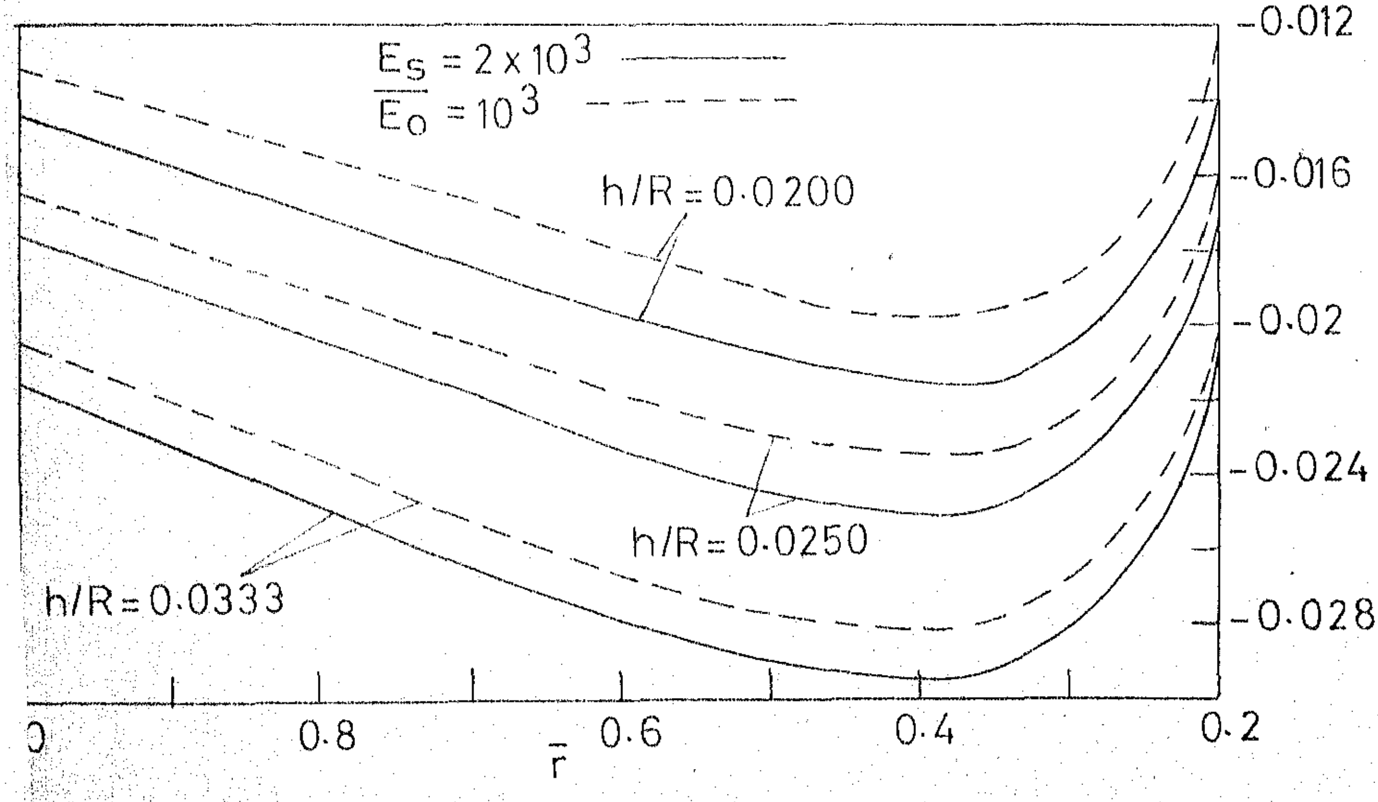


FIG. 3.3c SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES



IG. 3.3d SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{N}_1$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

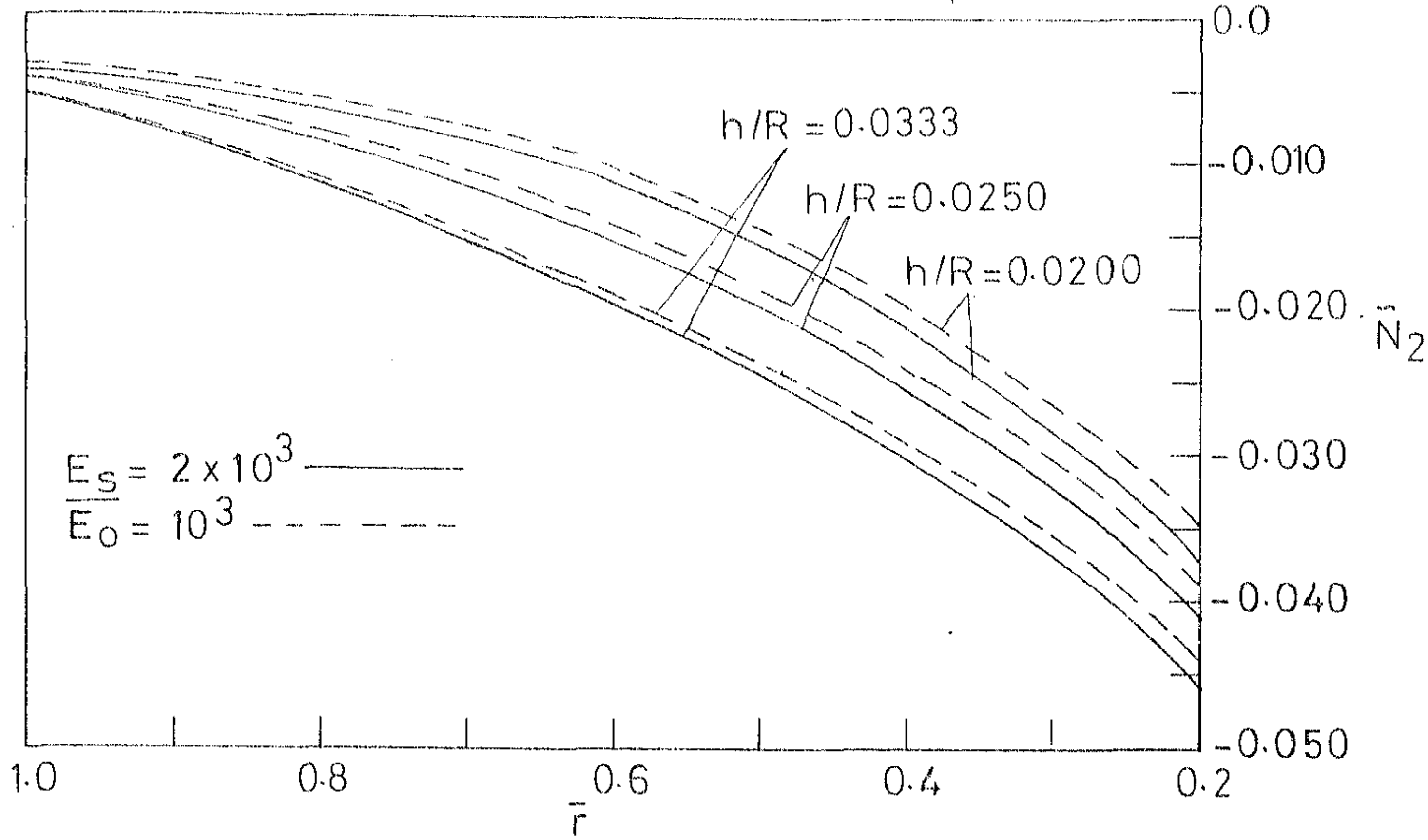


FIG. 3.3e SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\bar{N}_2$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

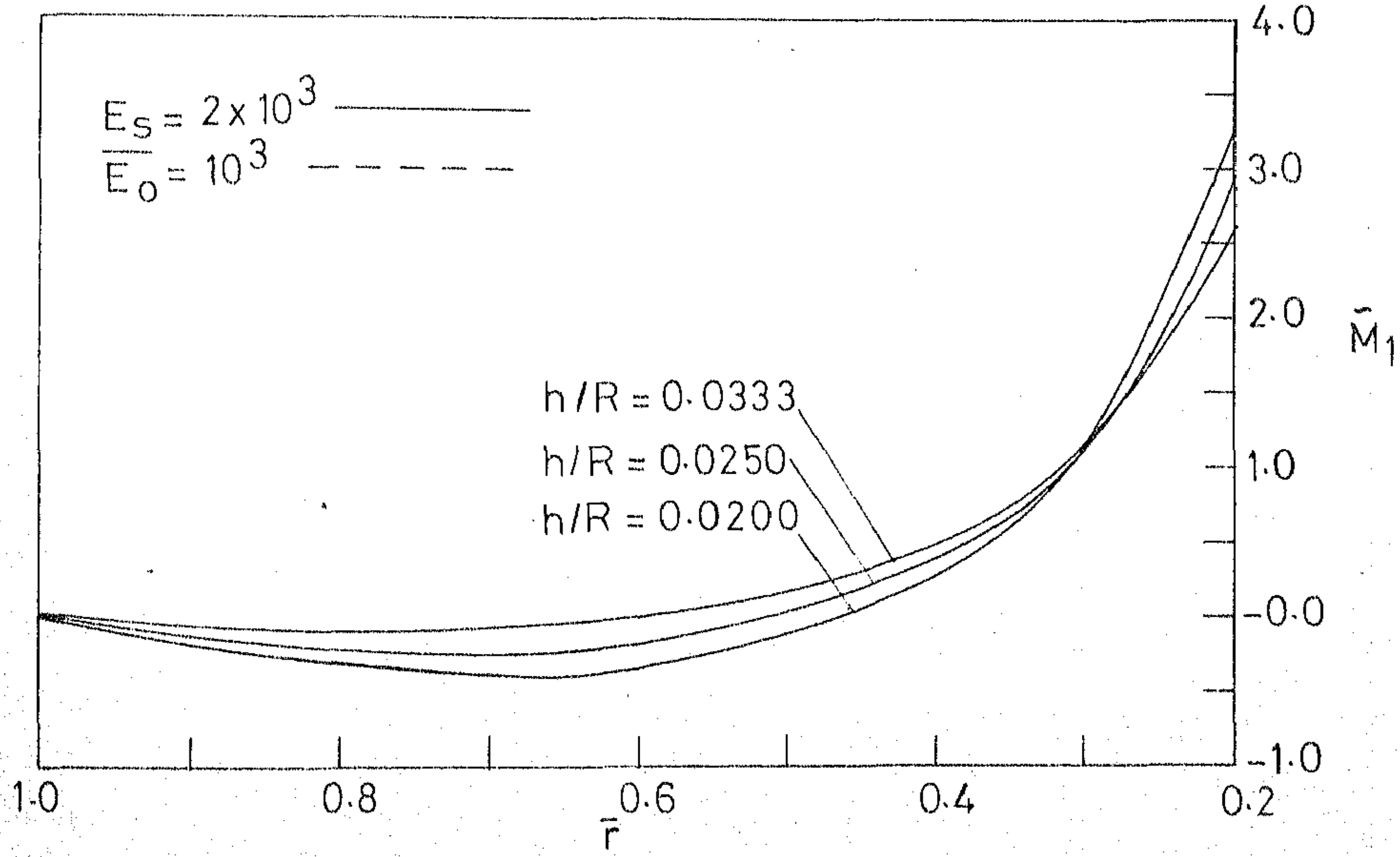


FIG. 3.3f SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\bar{M}_1$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

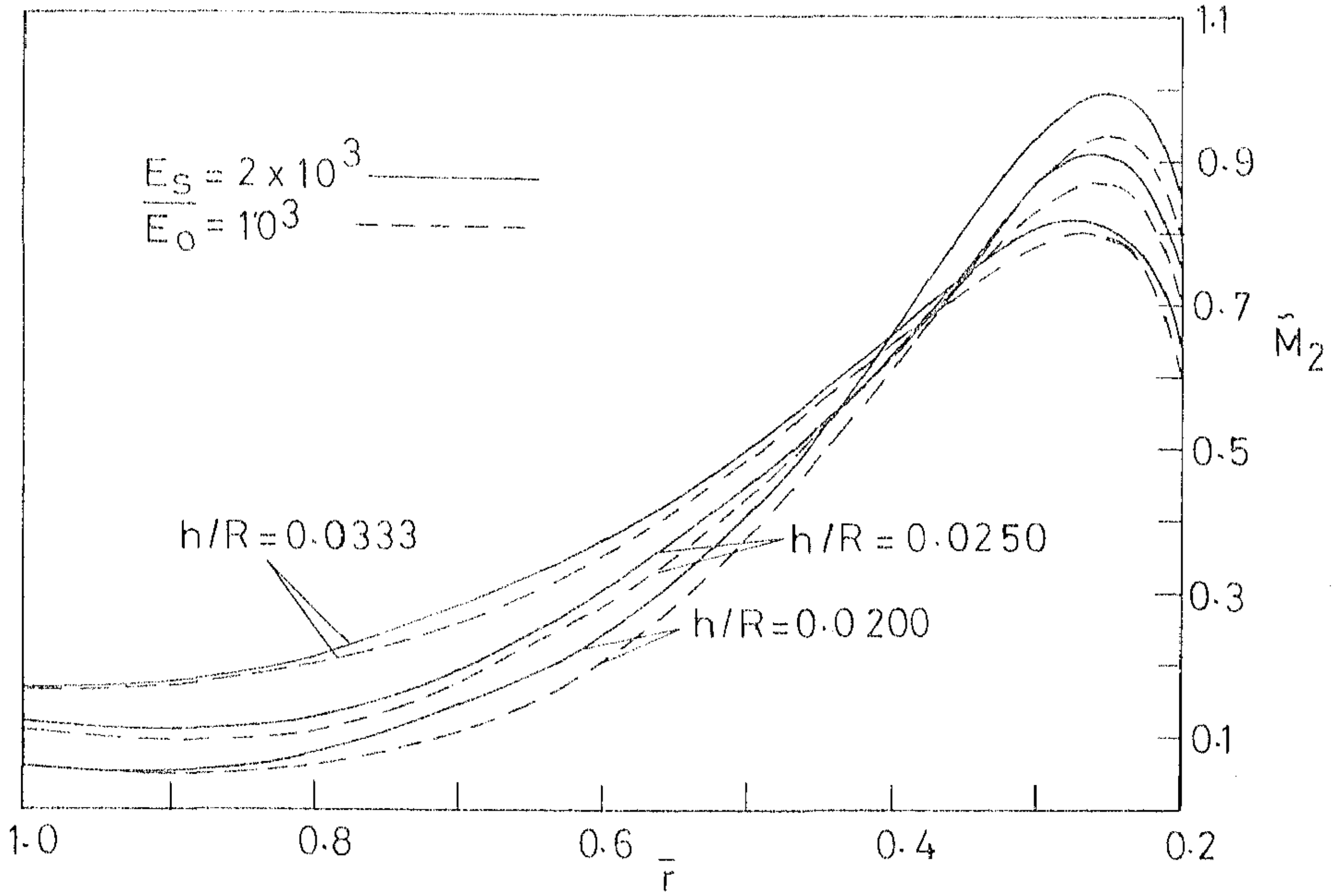


FIG. 3.3g SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{M}_2$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

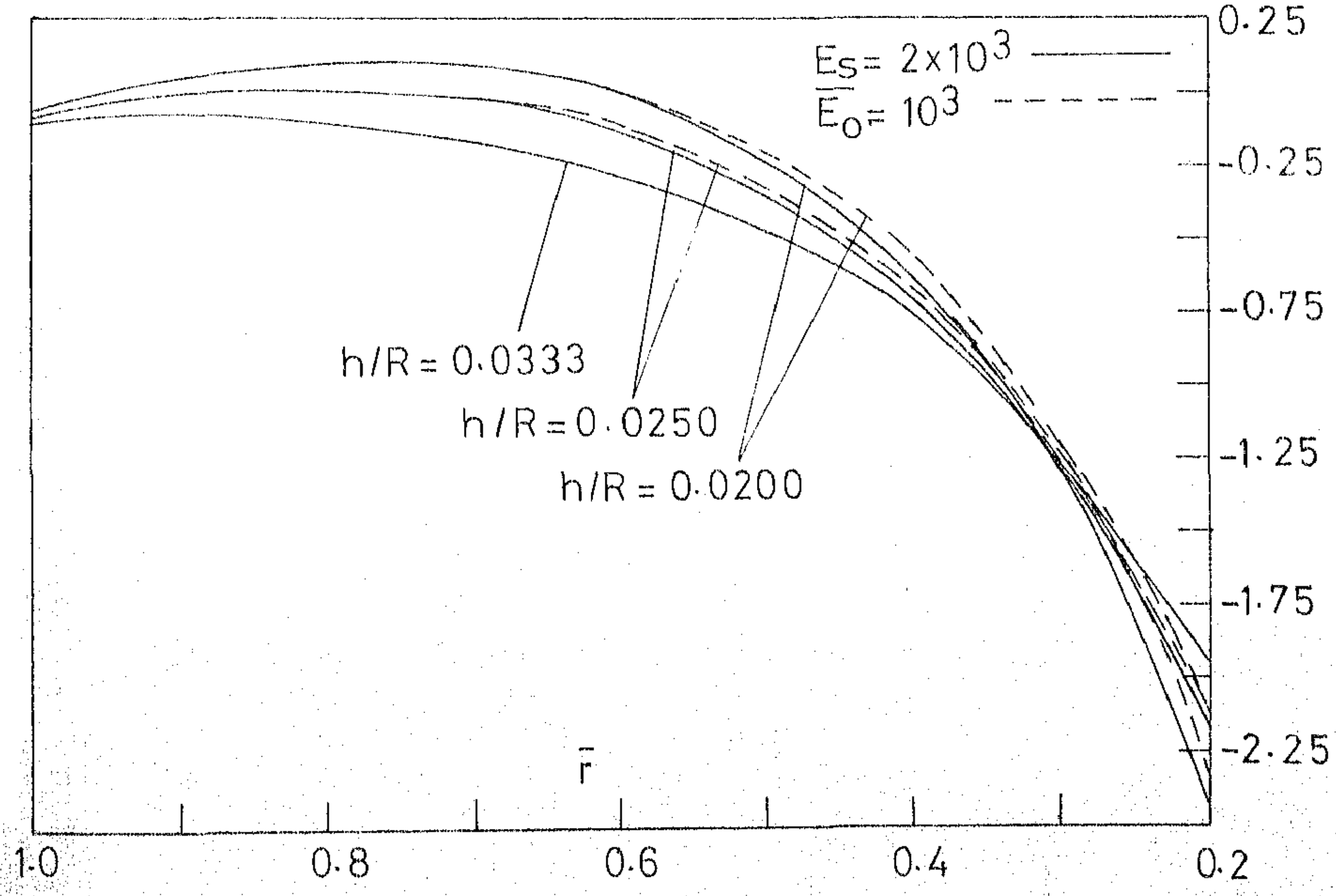


FIG. 3.3h SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{q}$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

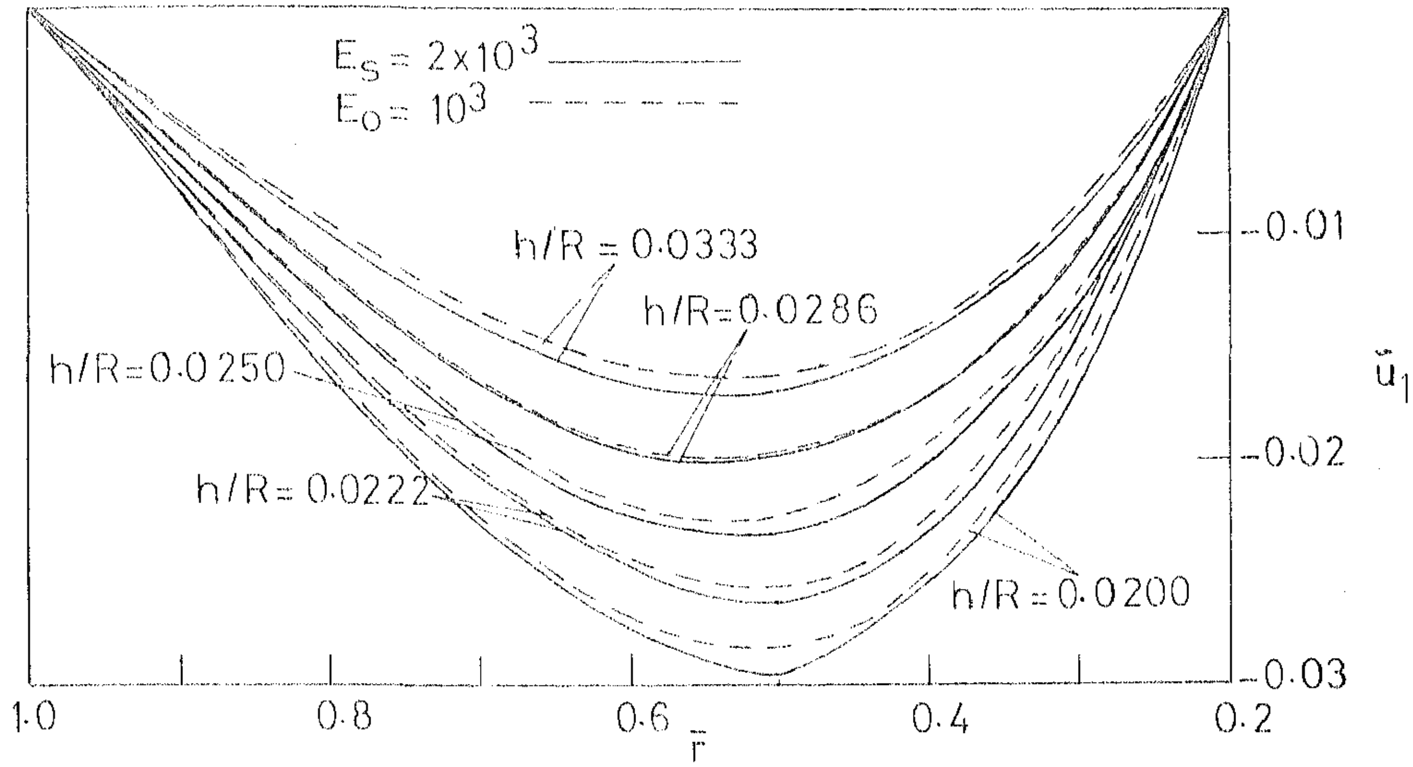


FIG. 3.4a FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{u}_1$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

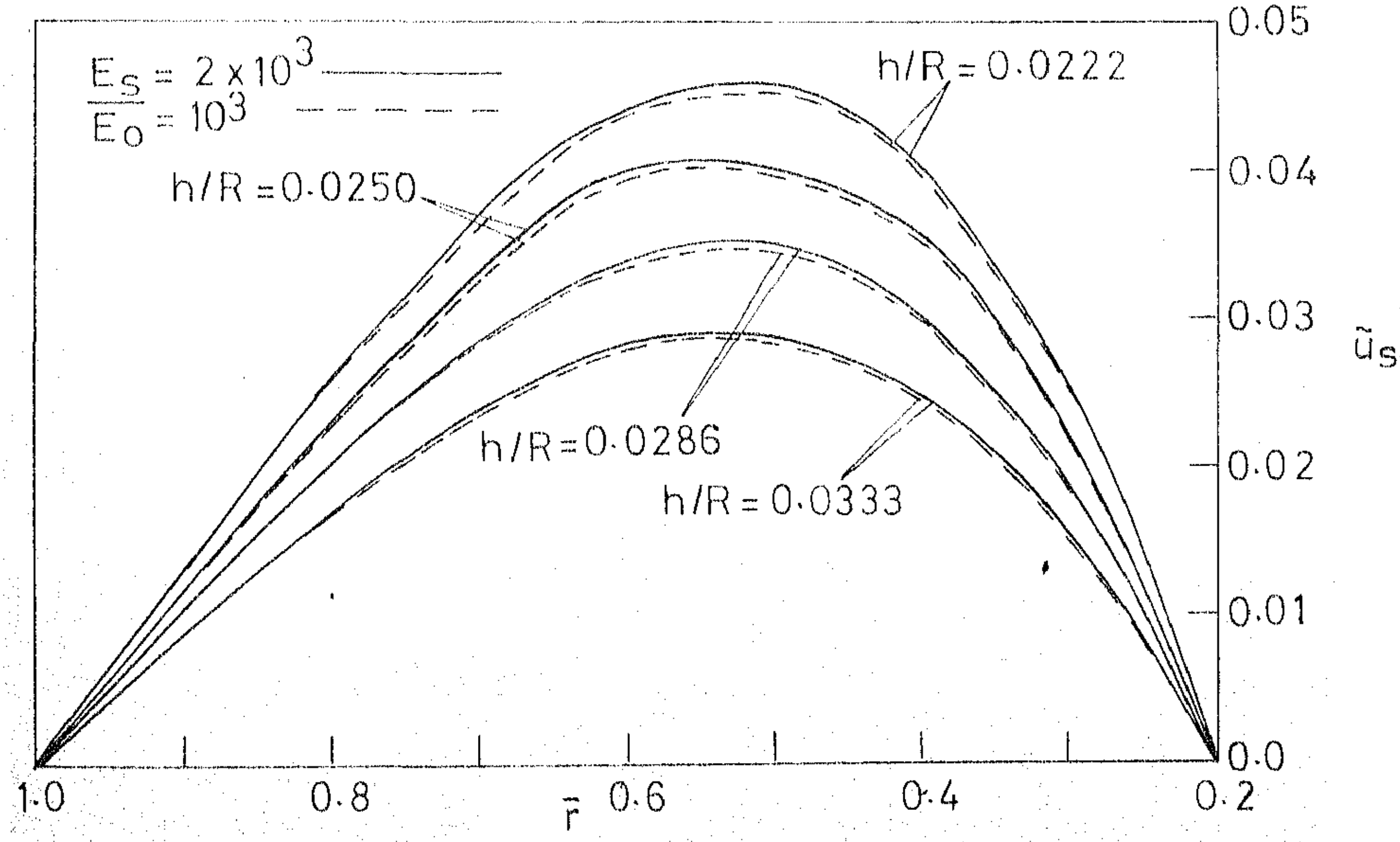


FIG. 3.4b FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{u}_s$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

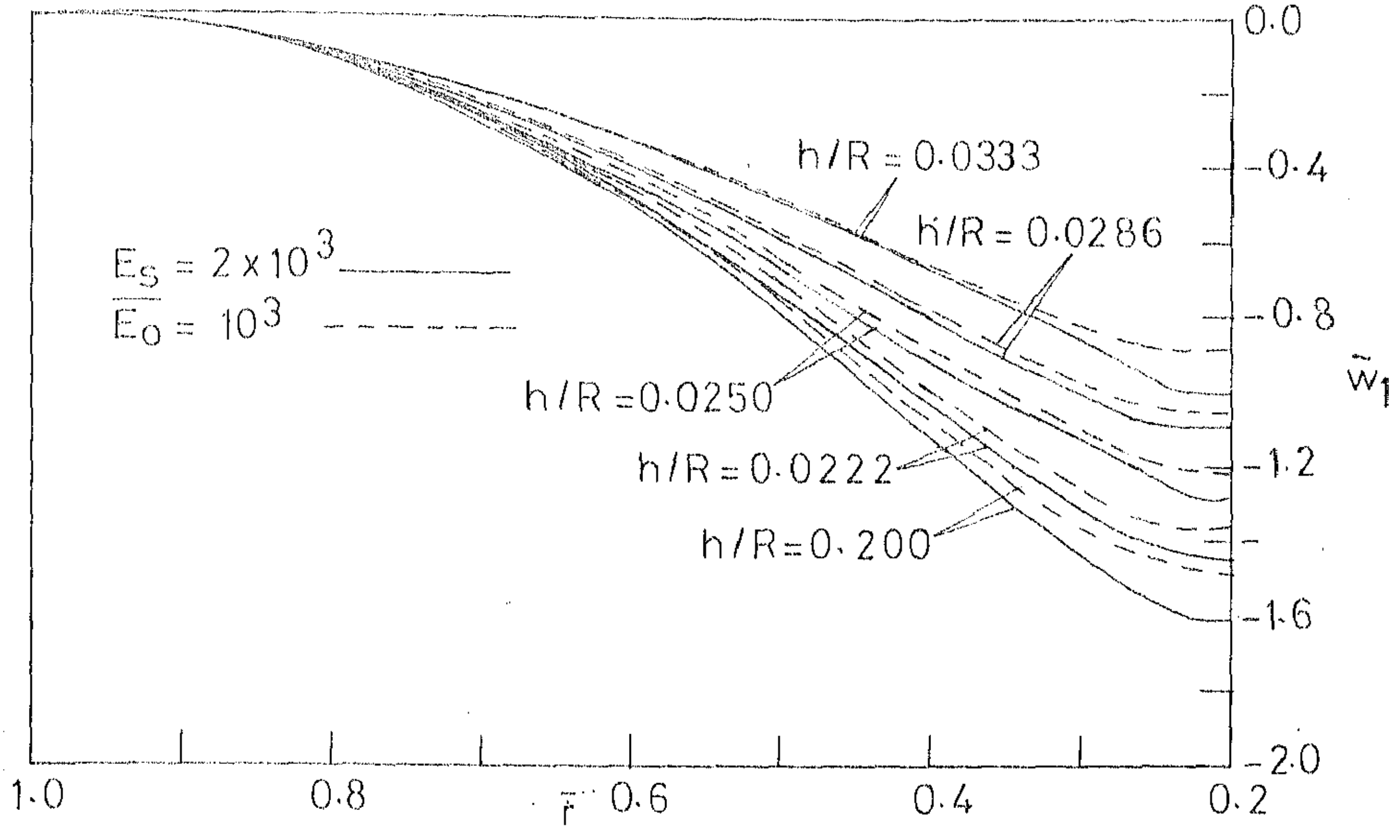


FIG. 3.4c FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

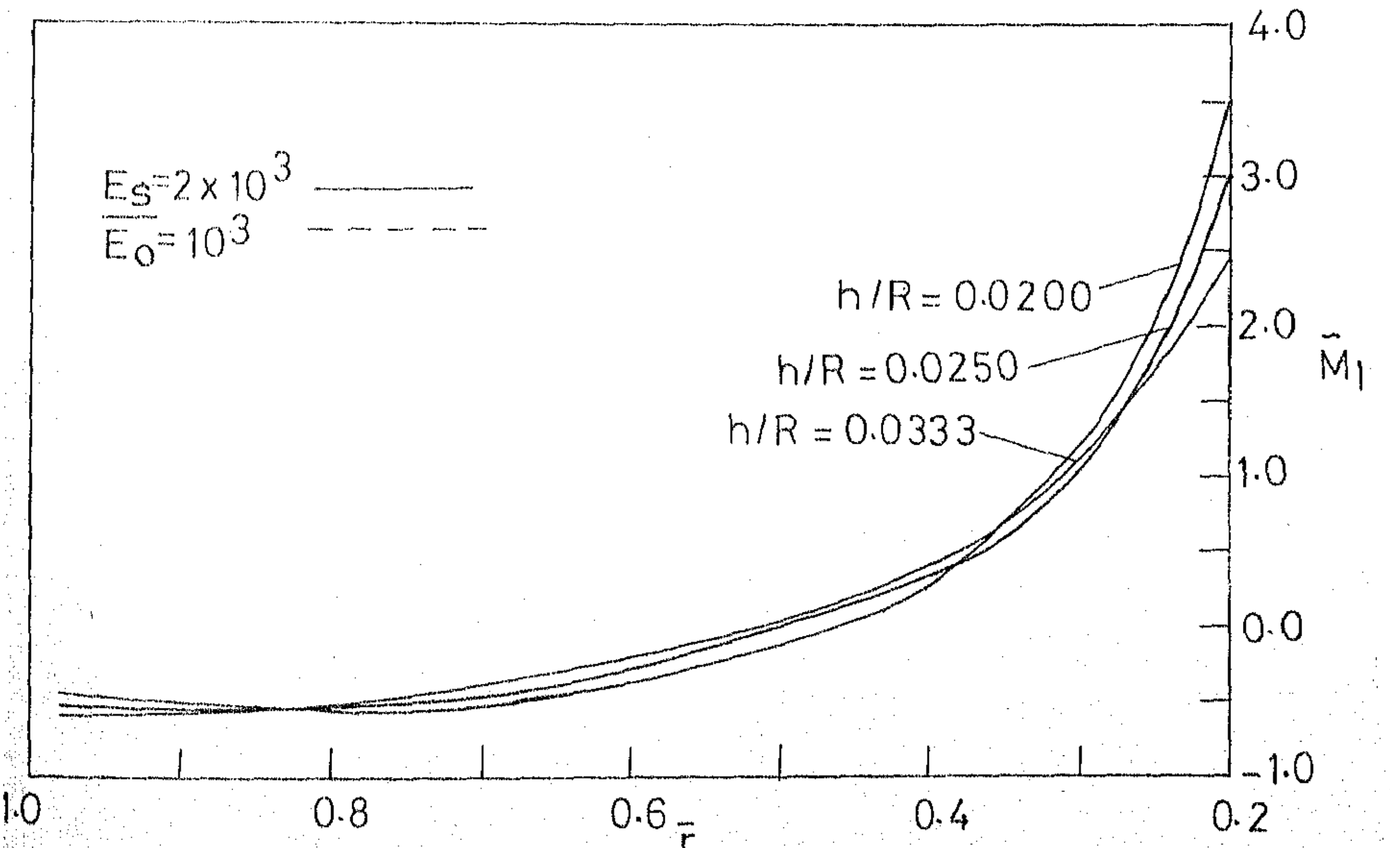


FIG. 3.4d FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_1$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

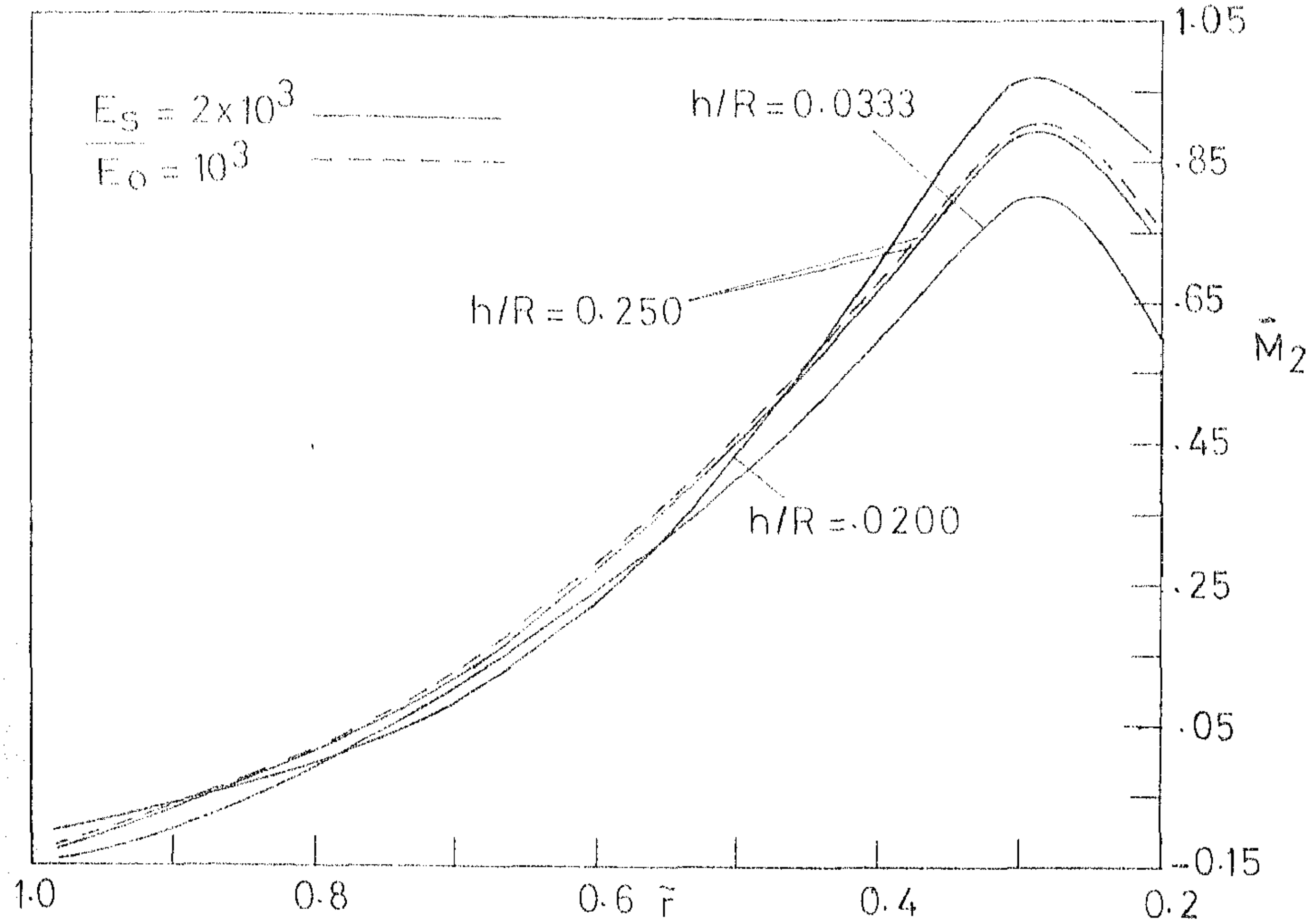


FIG. 3.4e FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_2$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

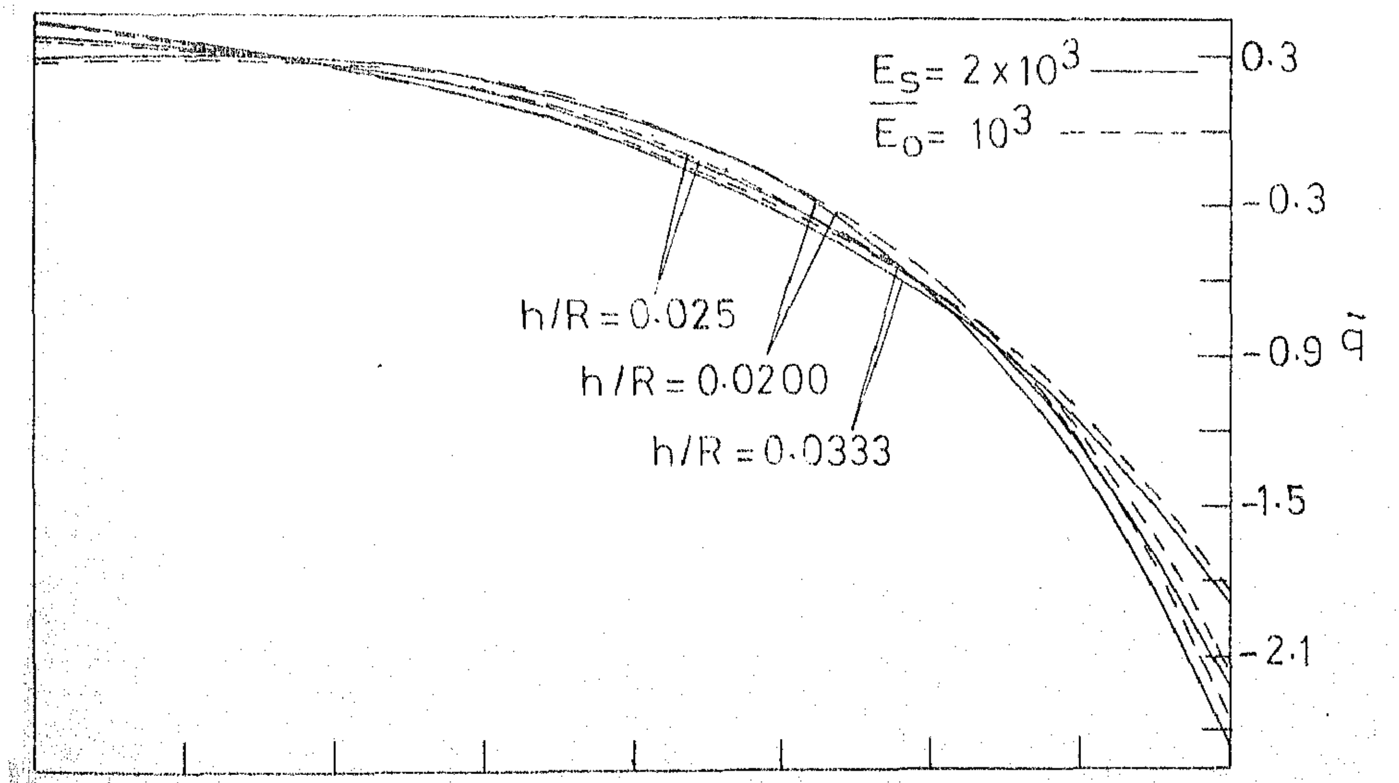


FIG. 3.4f FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (h/R) AND (E_s/E_0) VALUES

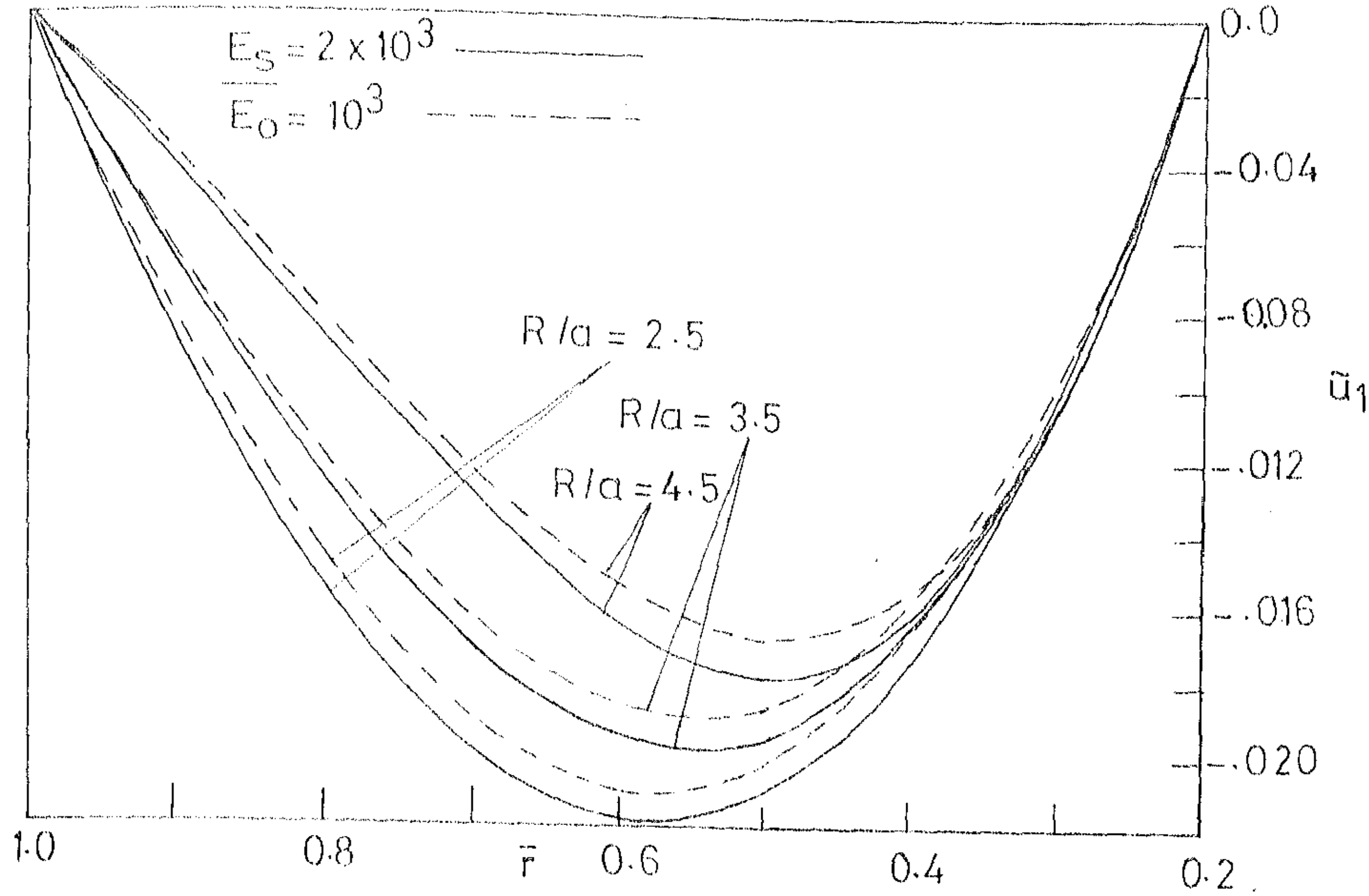


FIG. 3.5a SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{u}_1$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

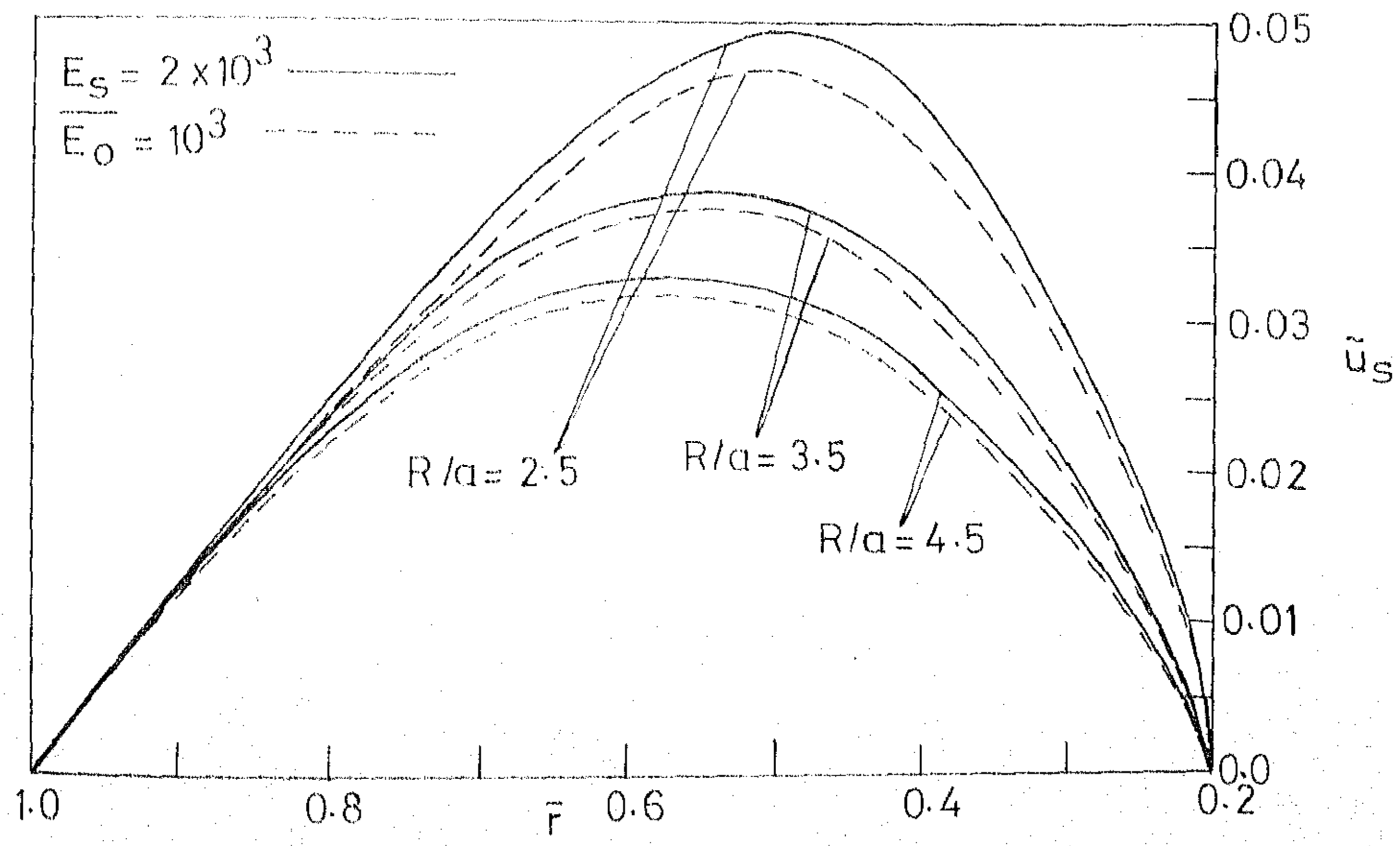


FIG. 3.5b SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{u}_s$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

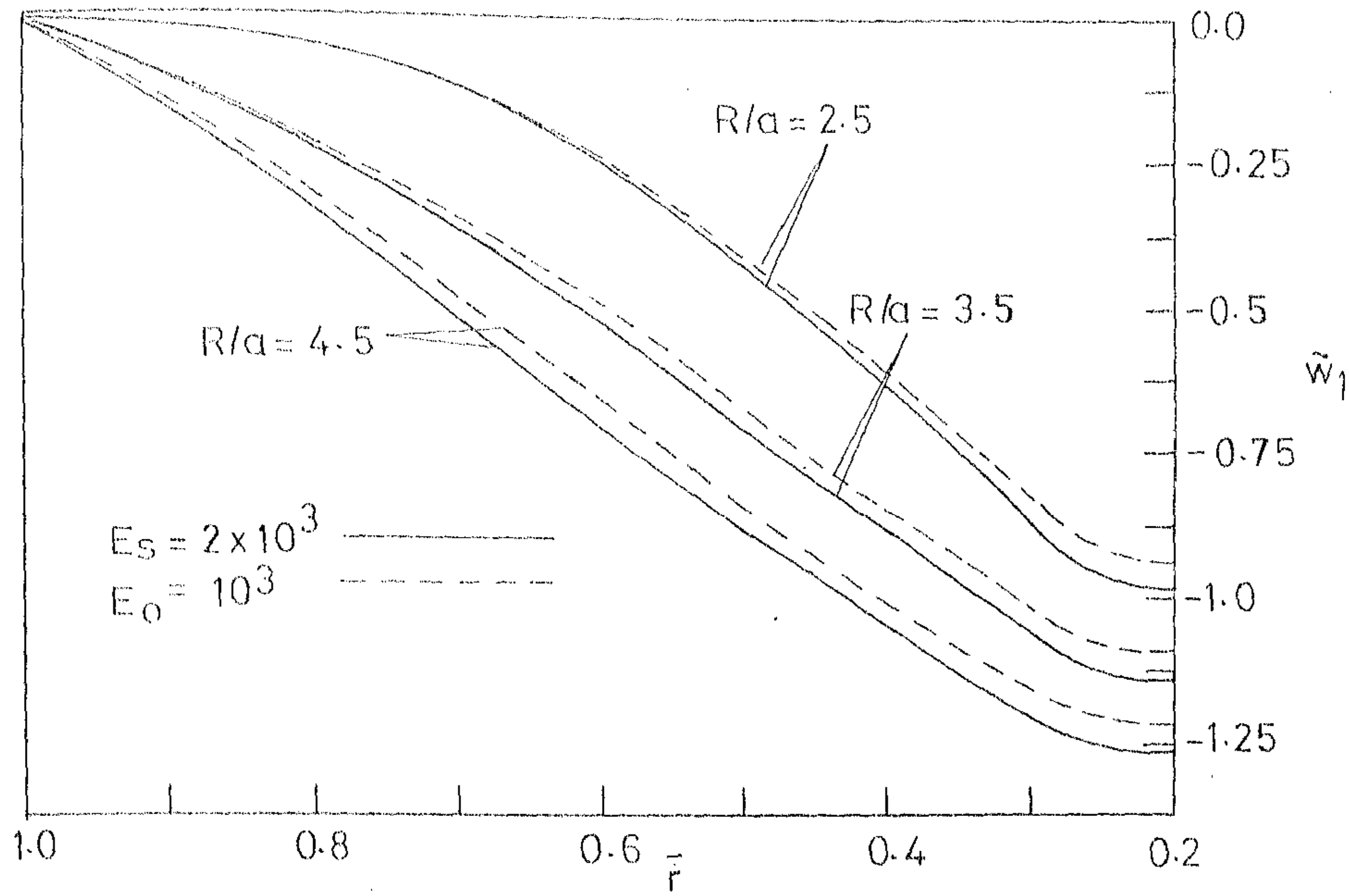


FIG. 3.5c SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{W}_1$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

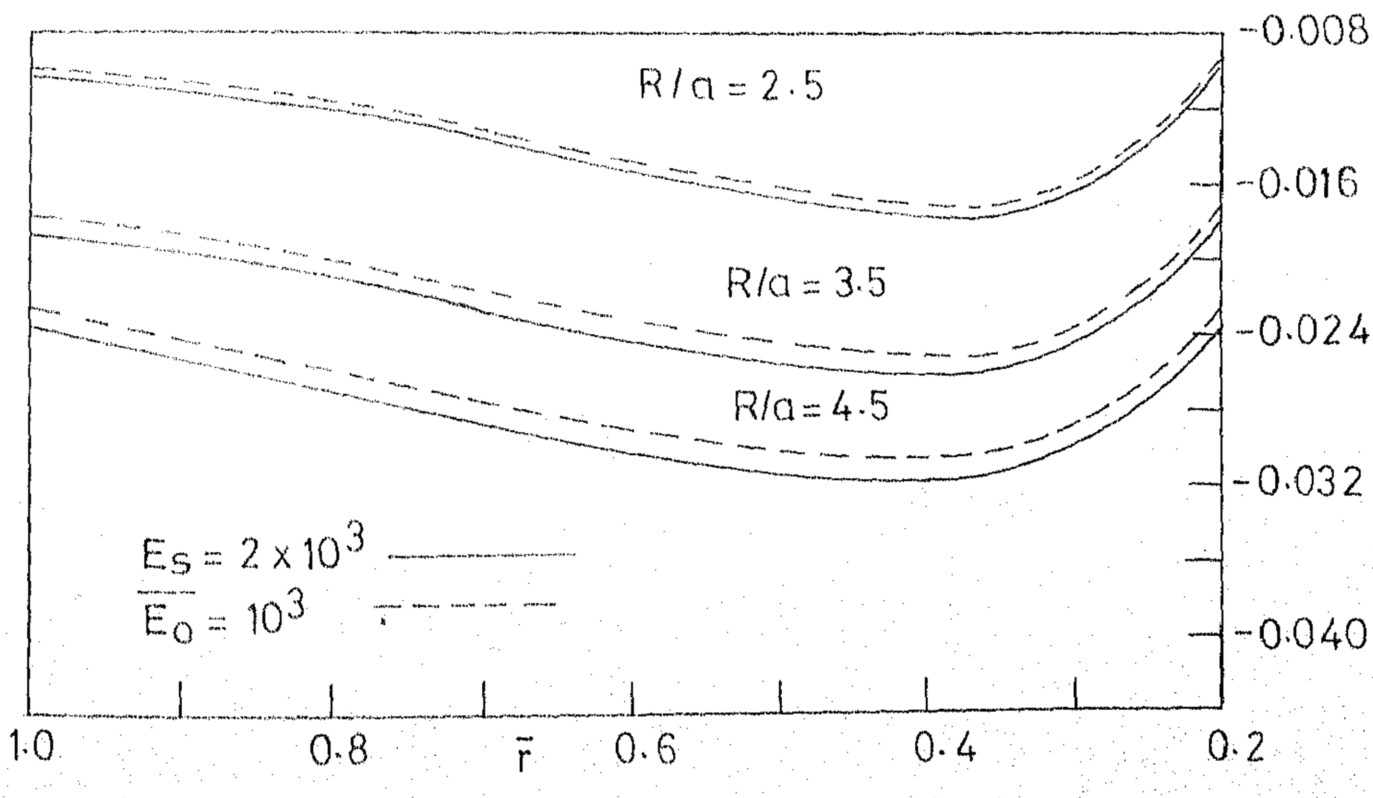


FIG. 3.5d SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{N}_1$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

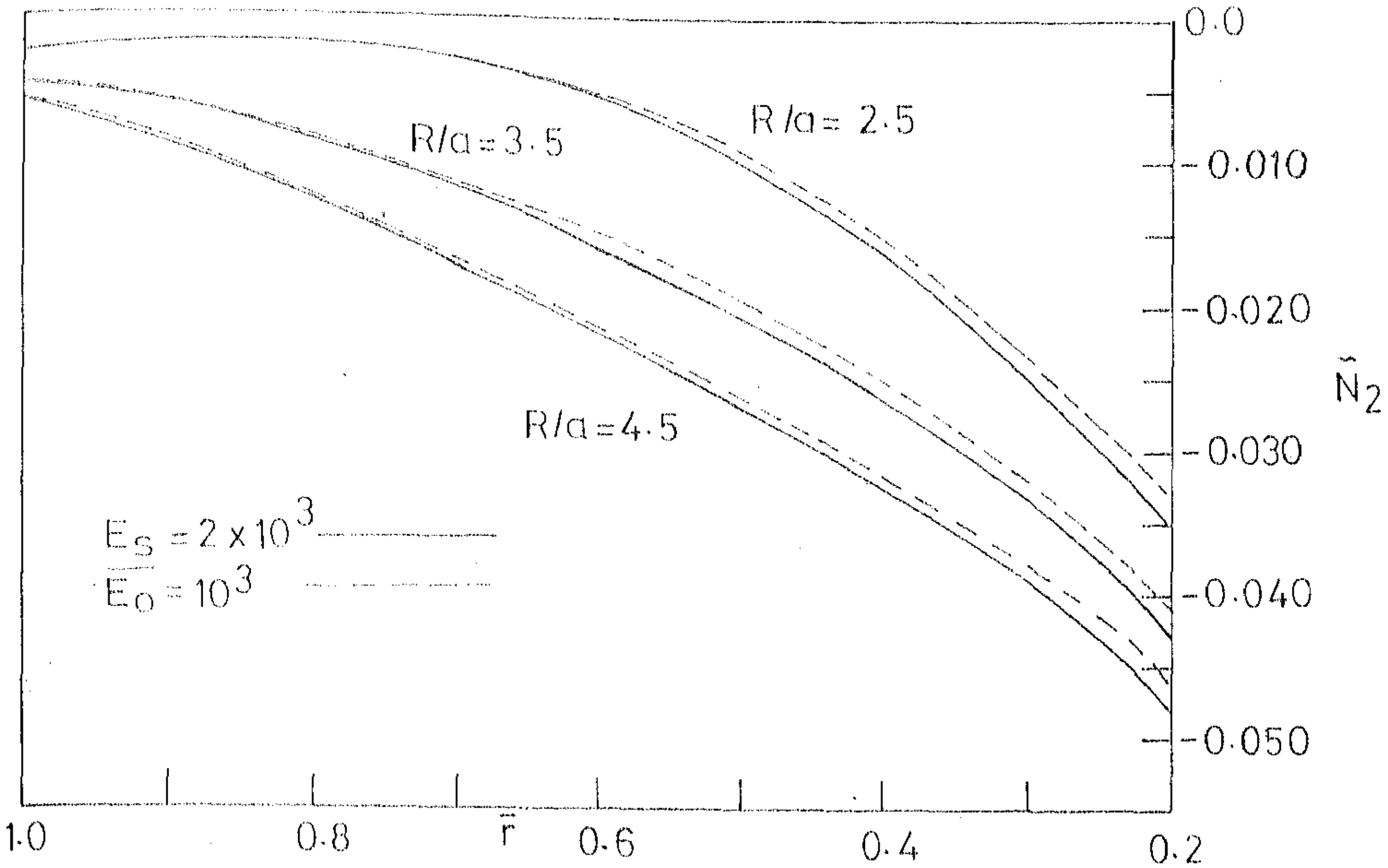


FIG. 3.5e SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\bar{N}_2$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

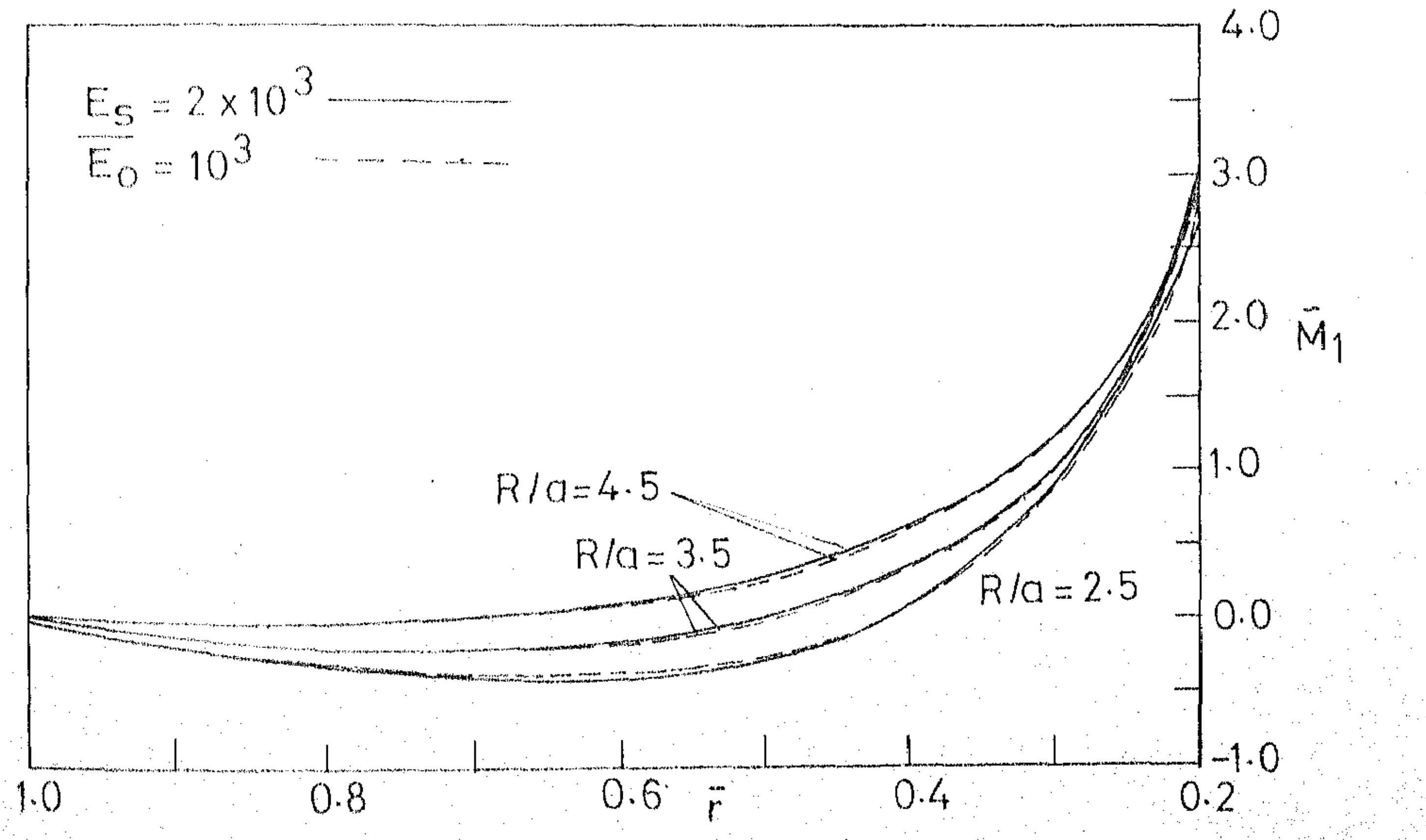


FIG. 3.5f SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\bar{M}_1$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

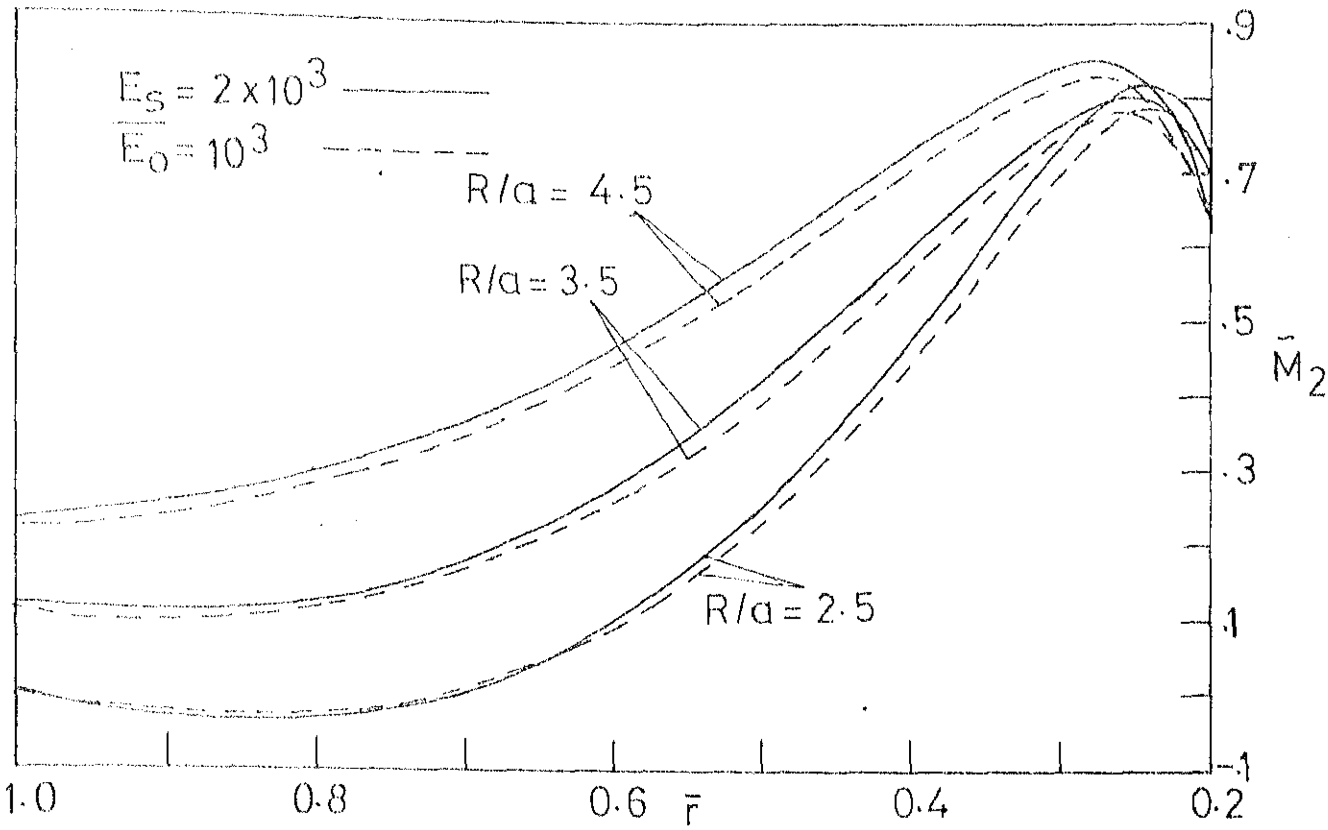


FIG. 3.5g SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_2$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

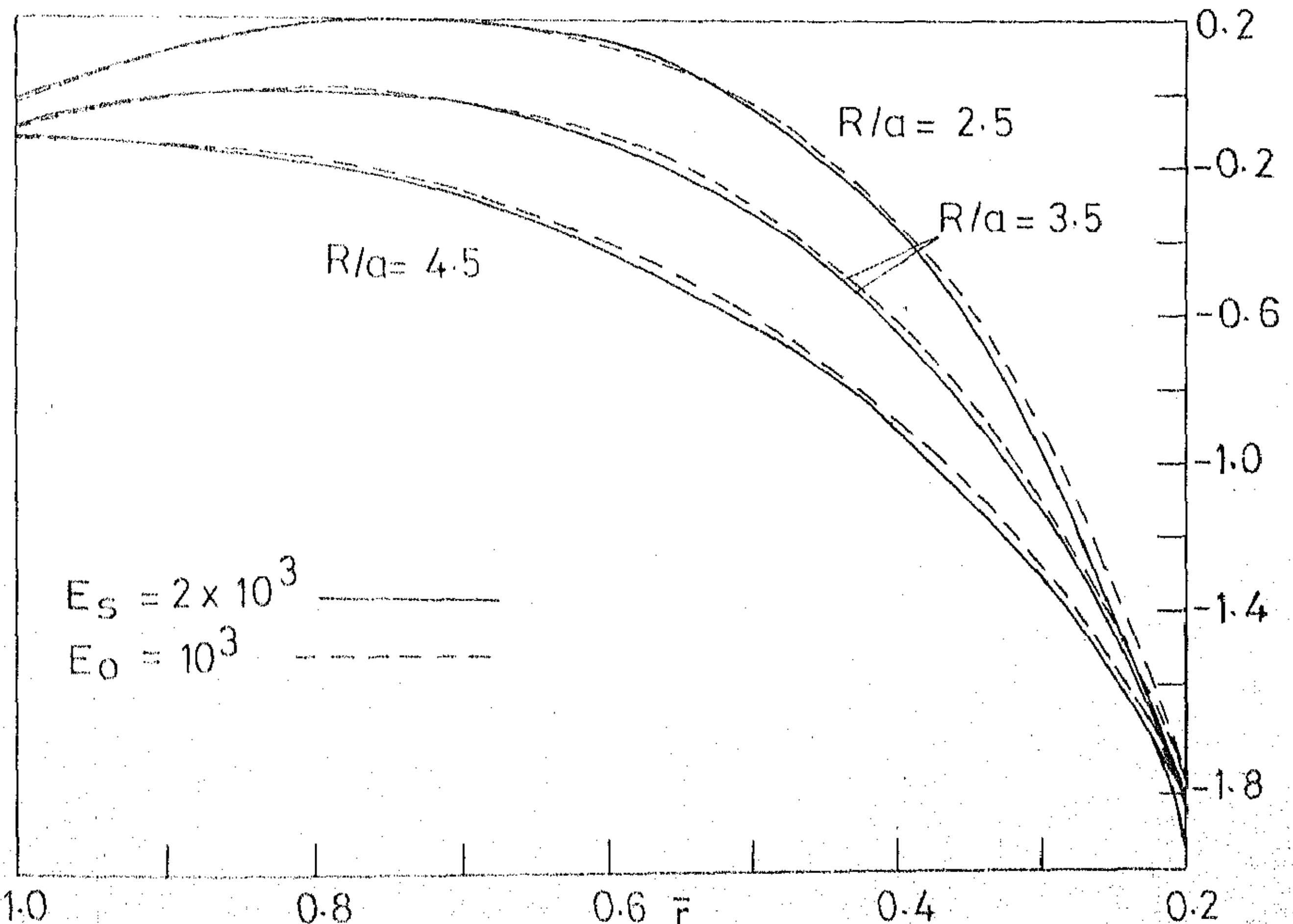


FIG. 3.5h SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

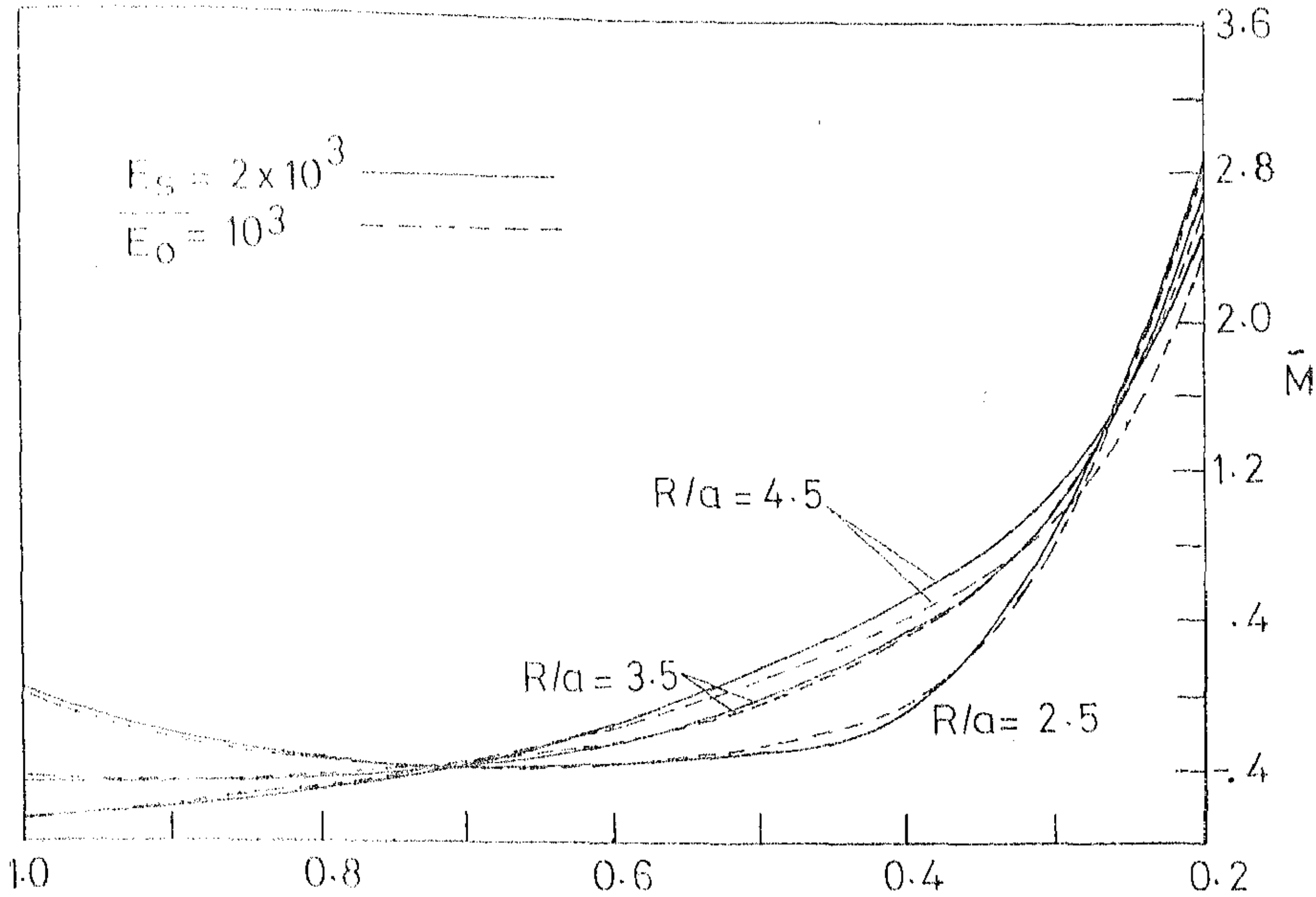


FIG. 3.6a FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\bar{M}_1$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

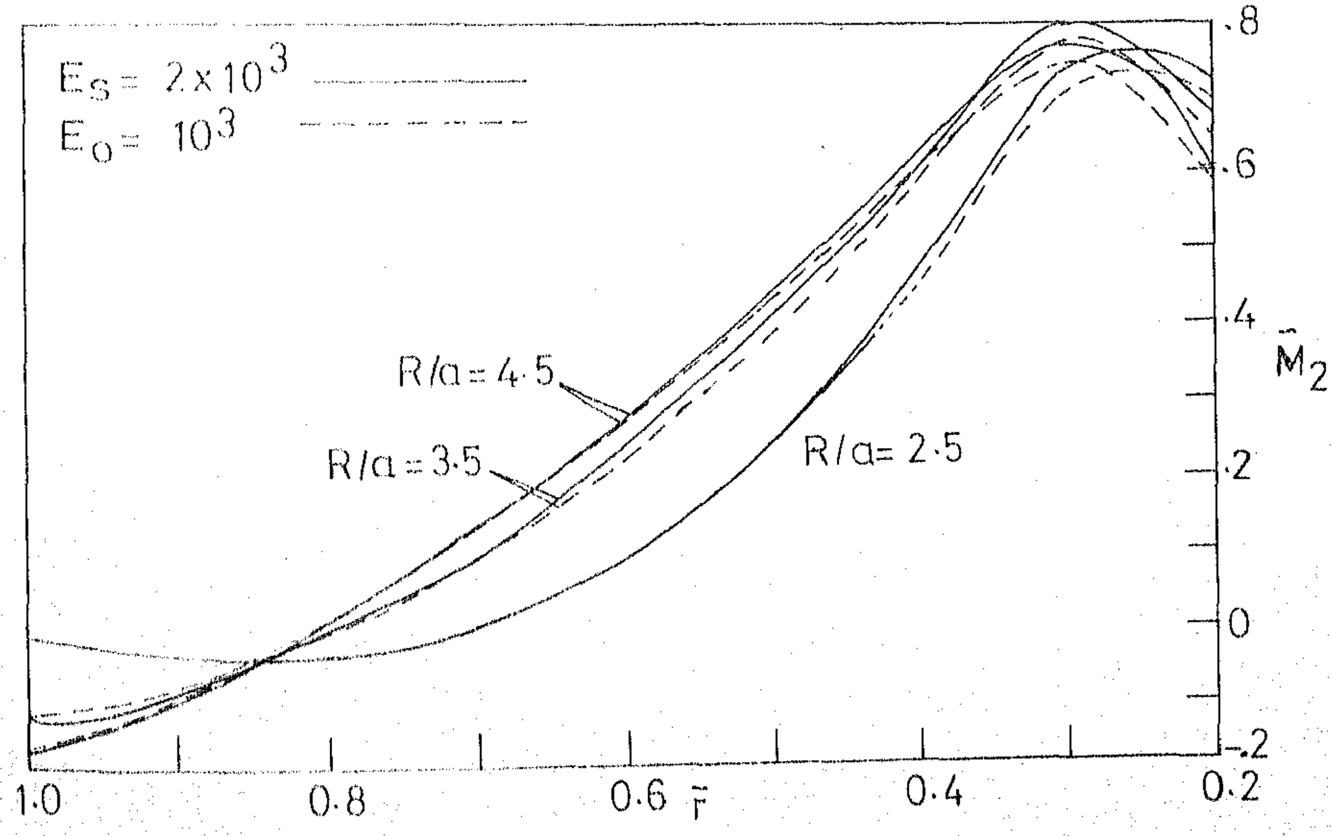


FIG. 3.6b FIXED SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\bar{M}_2$ FOR DIFFERENT (R/a) AND (E_s/E_0) VALUES

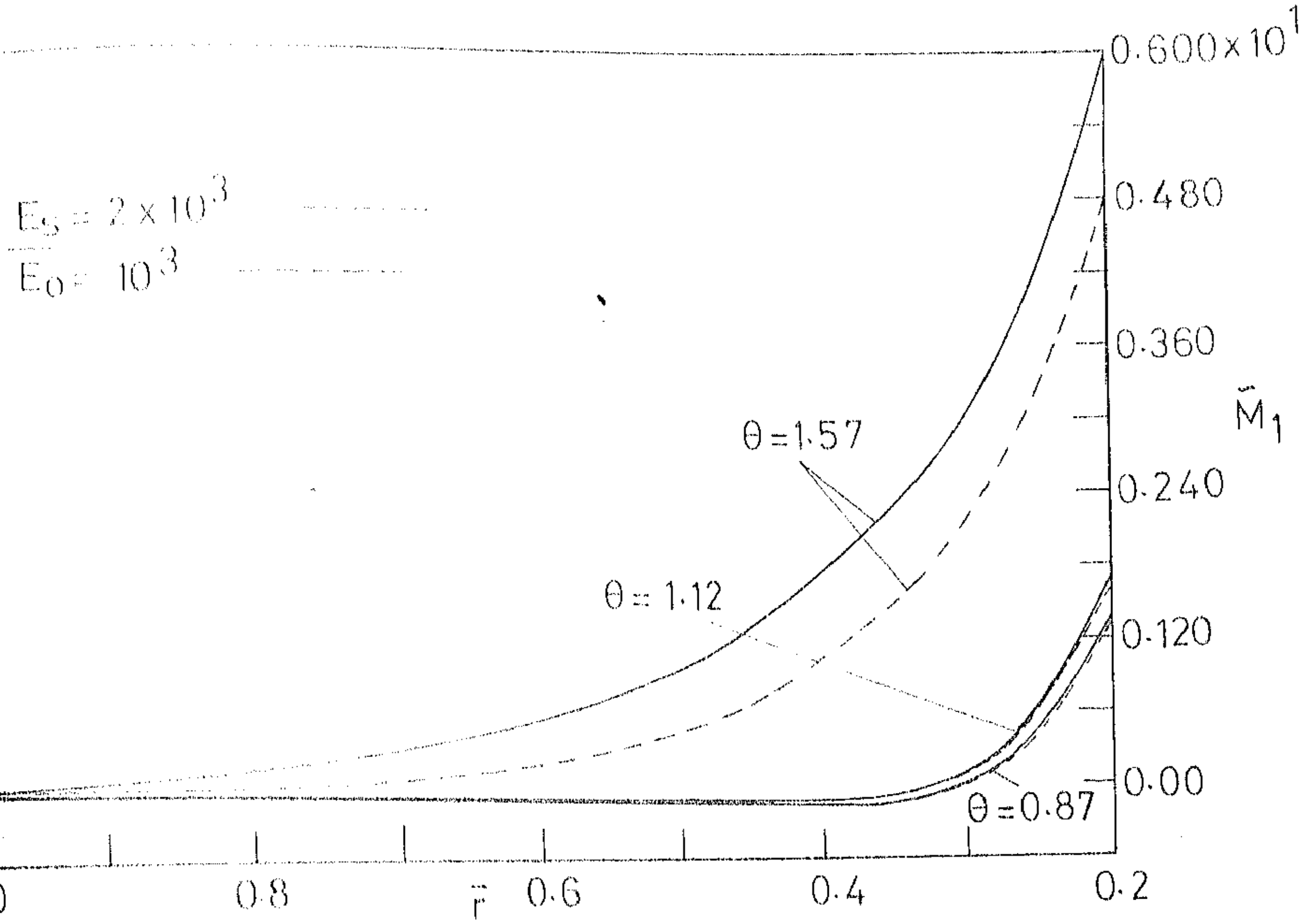


FIG. 3.7b SIMPLY SUPPORTED CONICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{M}_1$ FOR DIFFERENT θ AND (E_S/E_0) VALUES

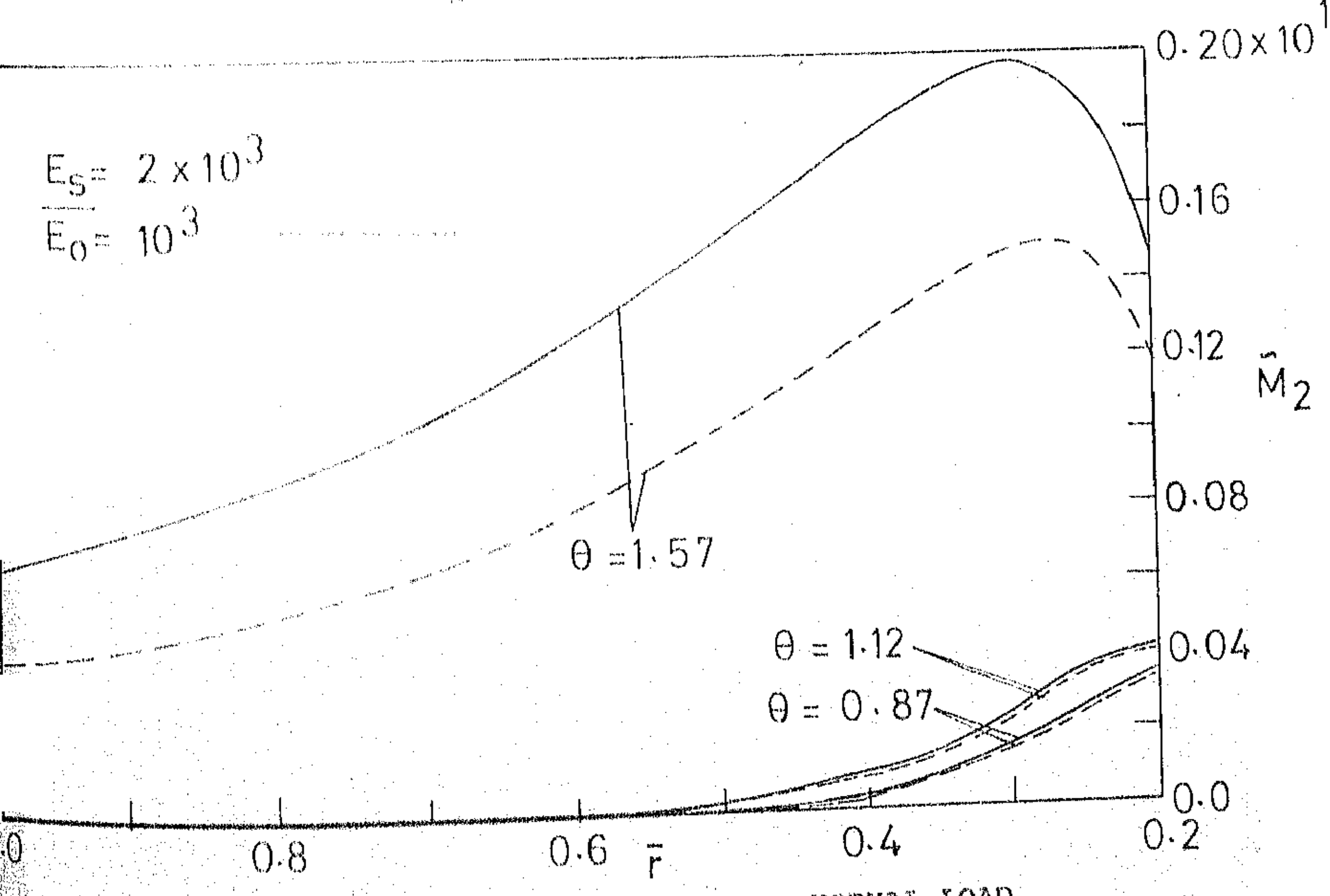


FIG. 3.7c SIMPLY SUPPORTED CONICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{M}_2$ FOR DIFFERENT θ AND (E_S/E_0) VALUES

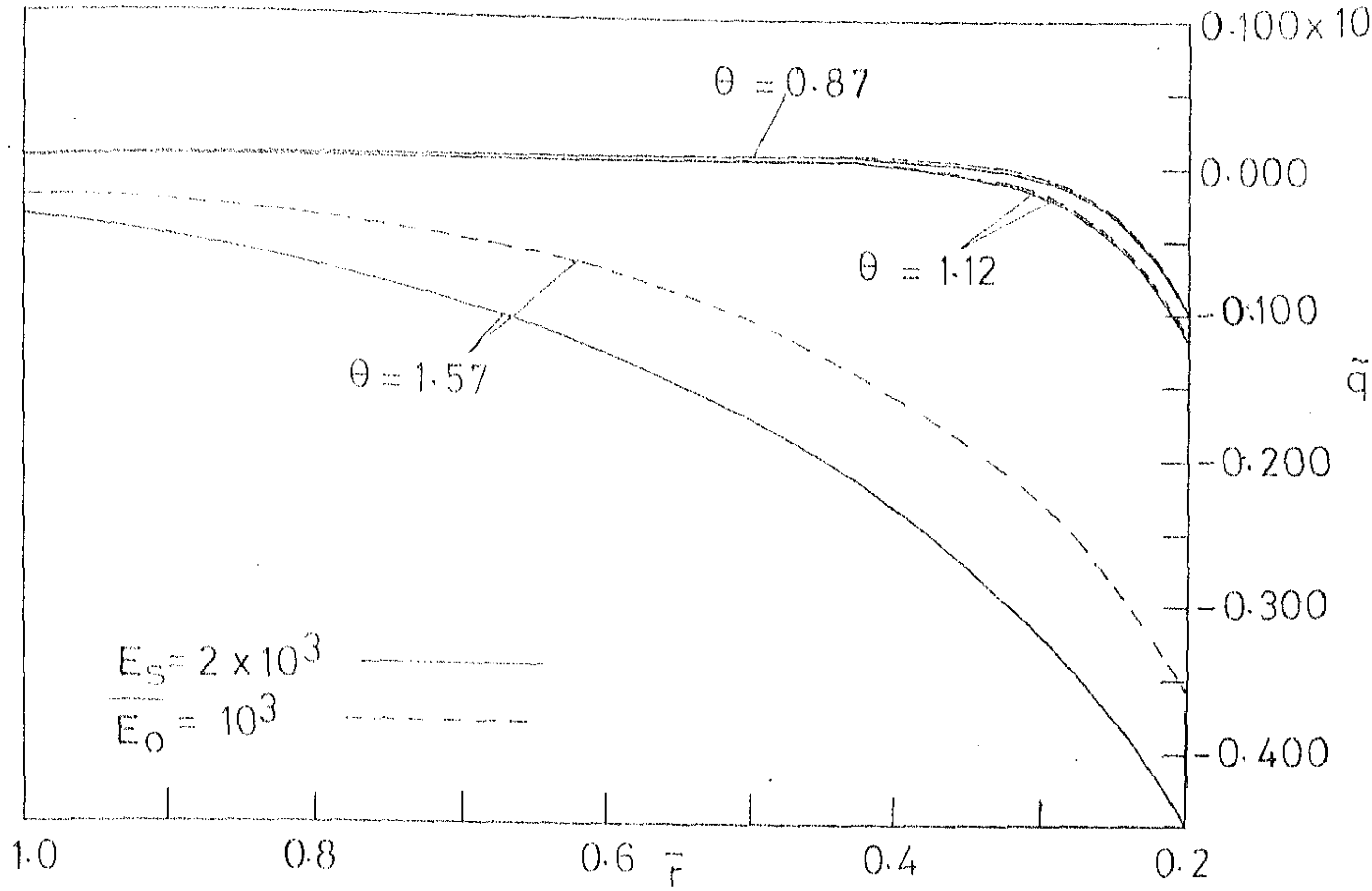


FIG. 3.7d SIMPLY SUPPORTED CONICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT θ AND (E_S/E_0) VALUES

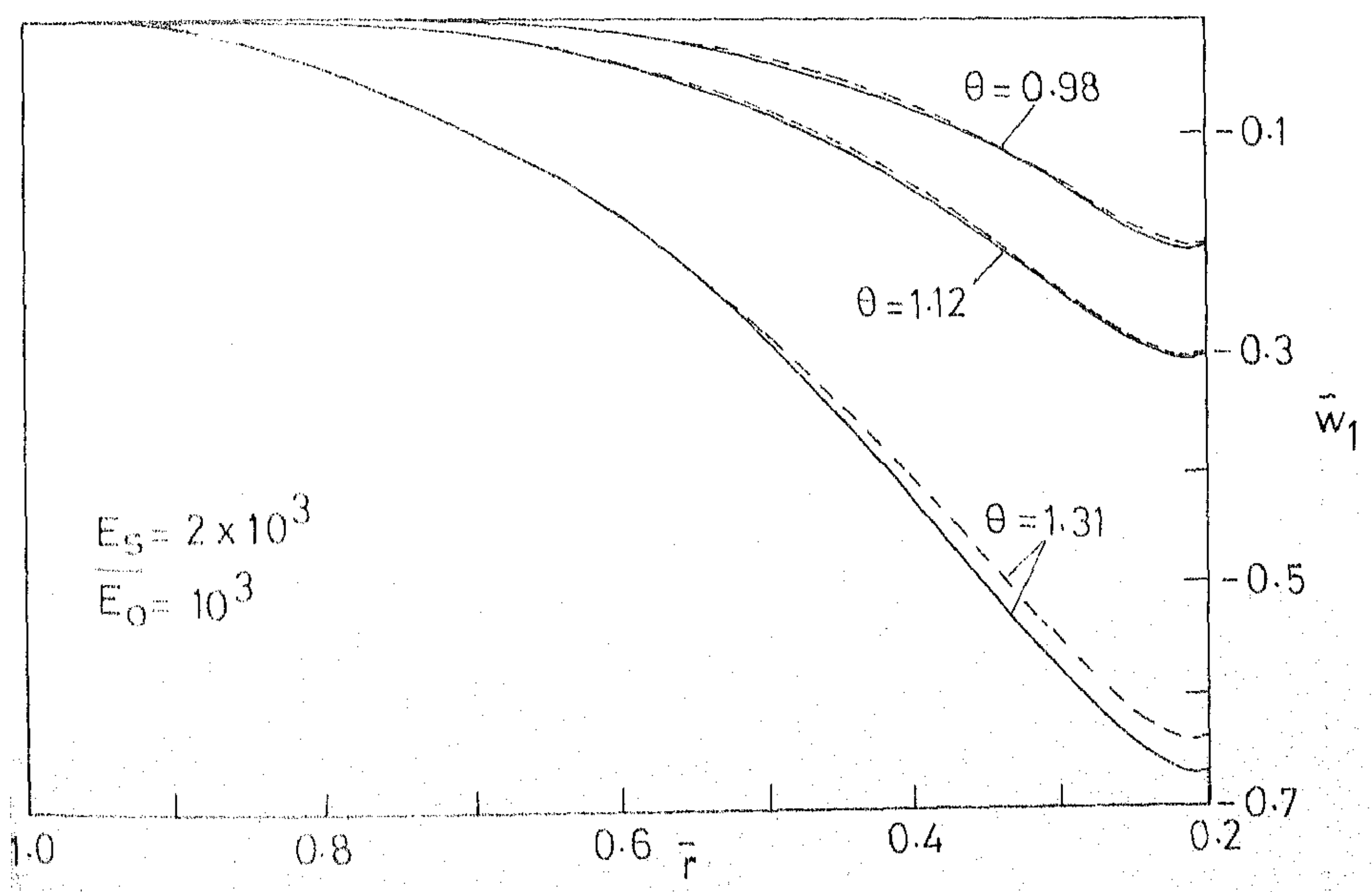


FIG. 3.8a FIXED CONICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{w}_1$ FOR DIFFERENT θ AND (E_S/E_0) VALUES

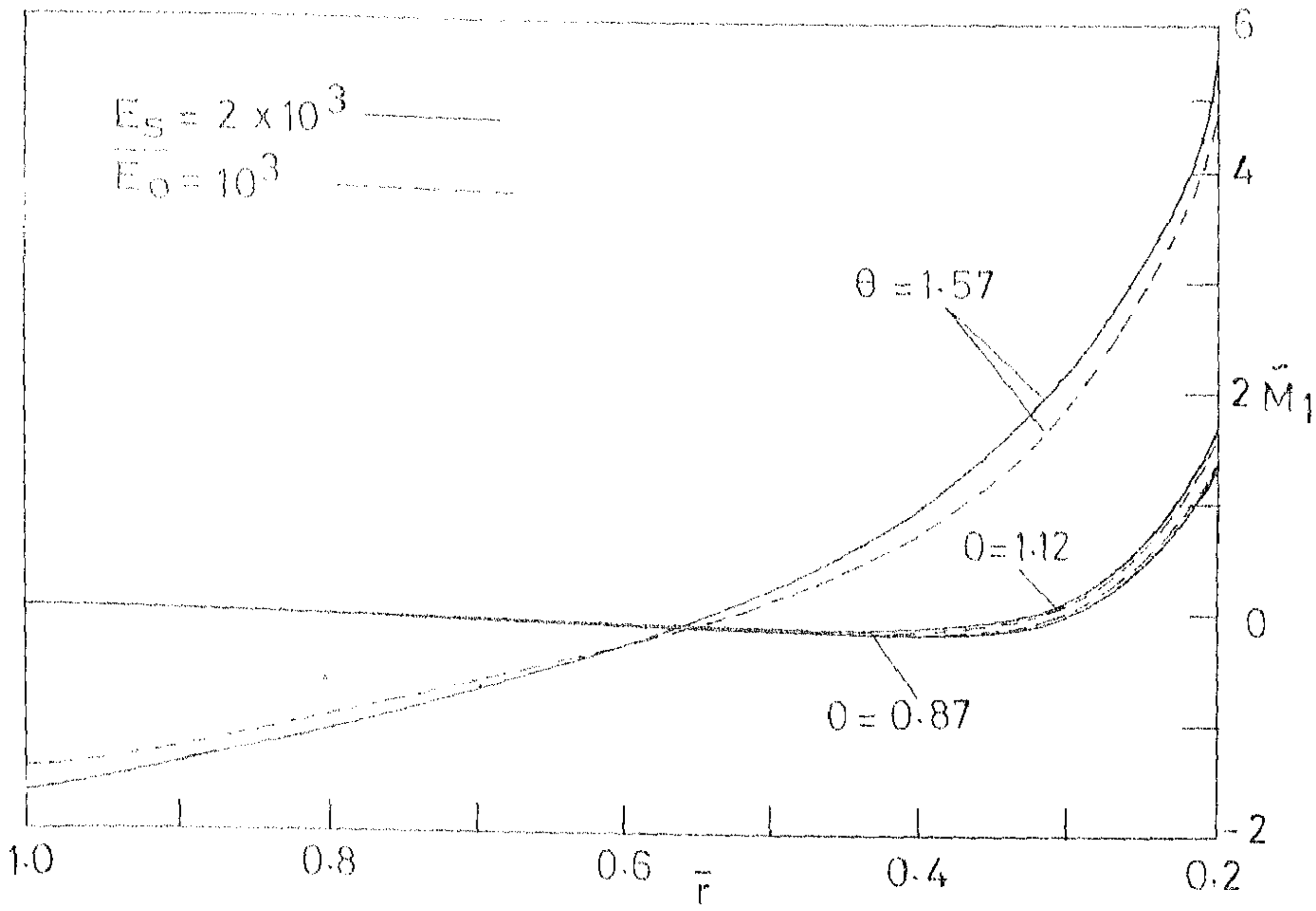


FIG. 3.8b FIXED CONICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_1$ FOR DIFFERENT θ AND (E_s/E_0) VALUES

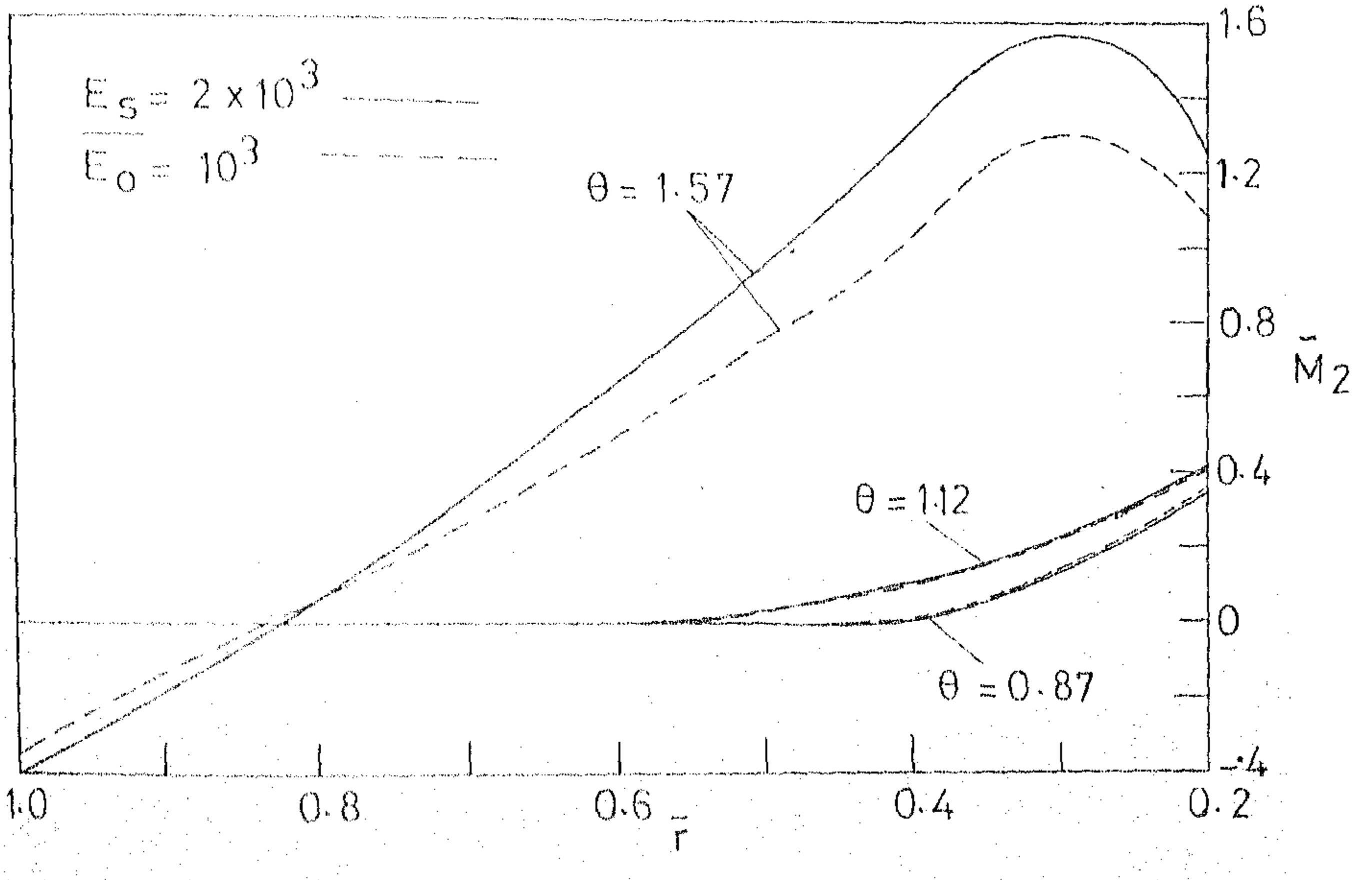


FIG. 3.8c FIXED CONICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_2$ FOR DIFFERENT θ AND (E_s/E_0) VALUES

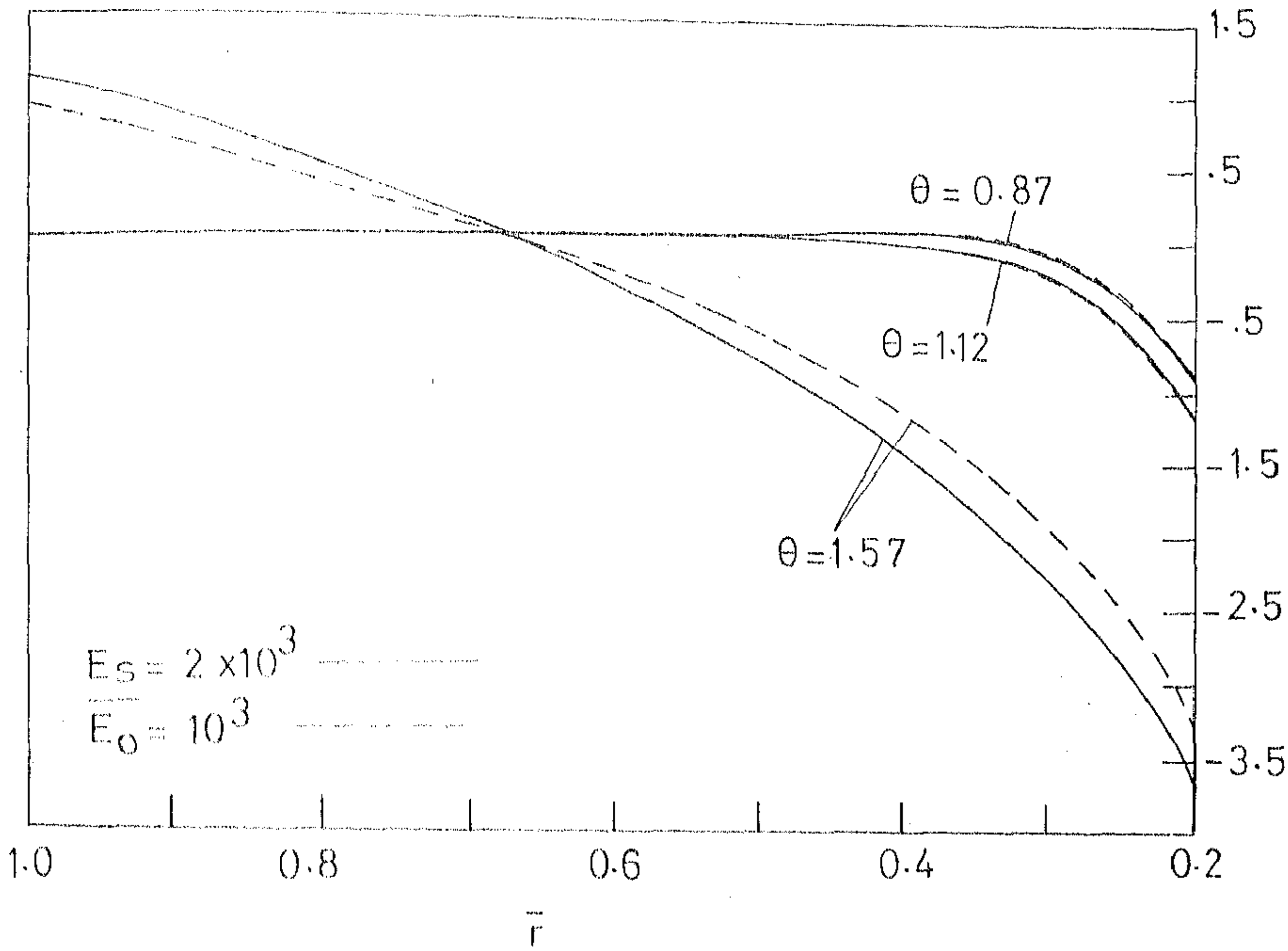


FIG. 3.8d FIXED CONICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT θ AND (E_s/E_0) VALUES

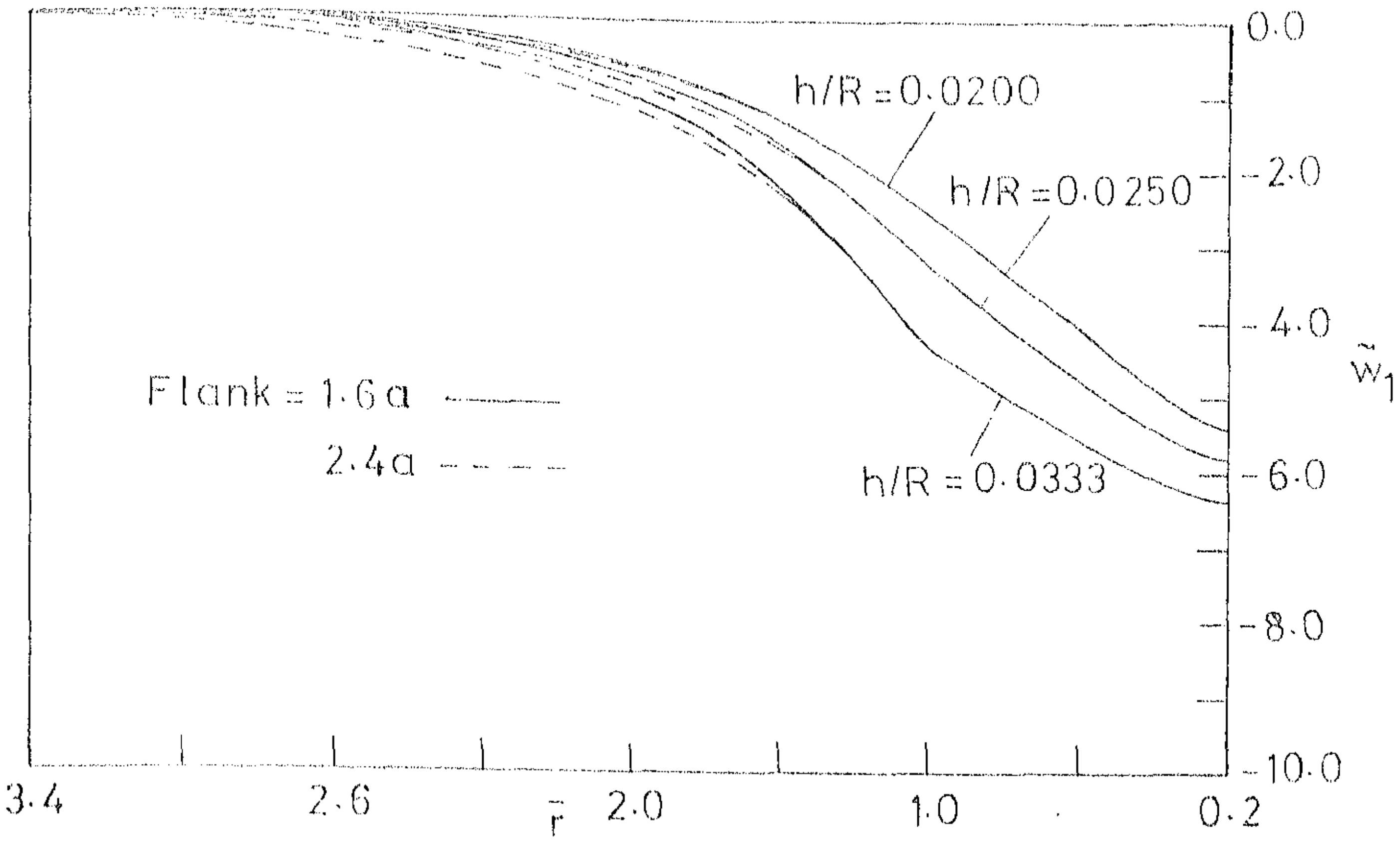


FIG. 3.9a FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (h/R) VALUES

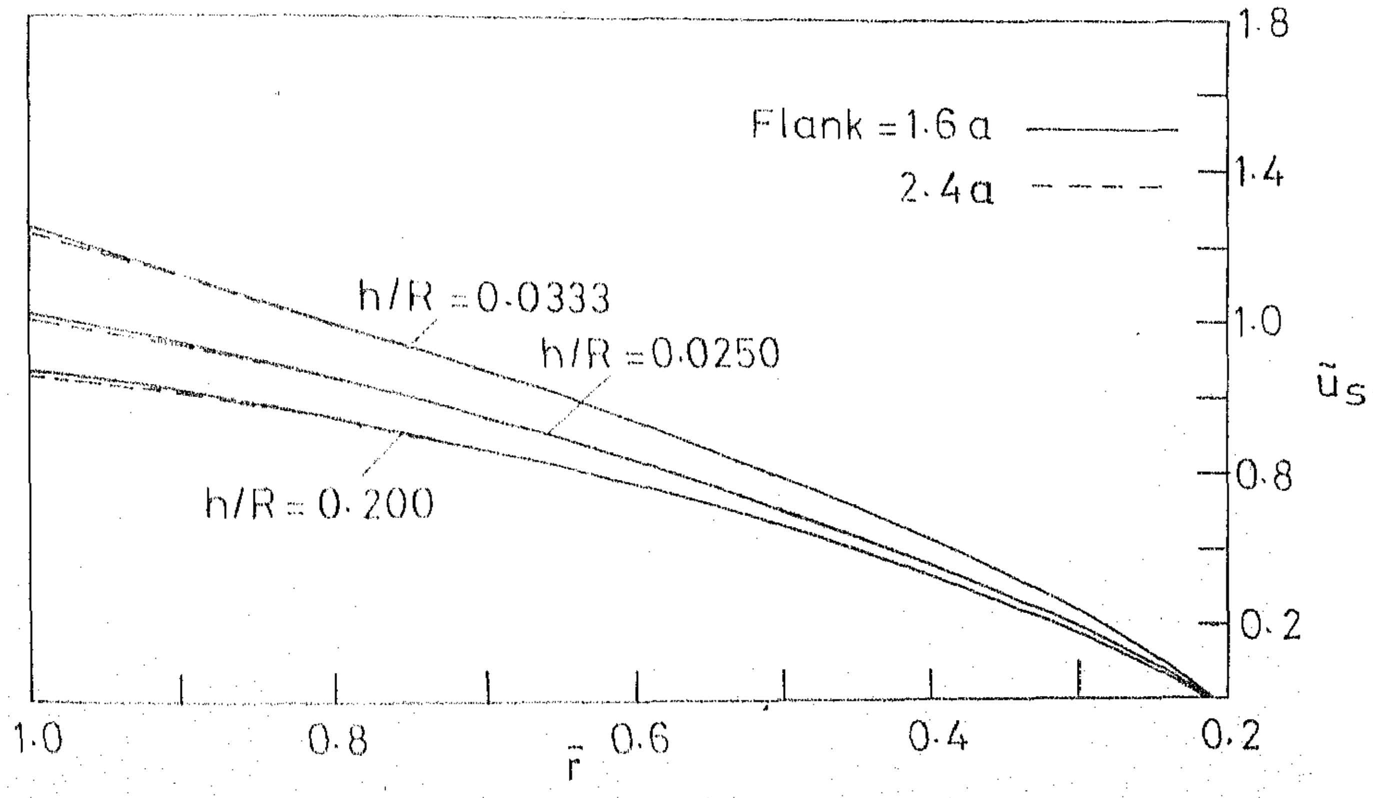


FIG. 3.9b FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{u}_s$ FOR DIFFERENT (h/R) VALUES

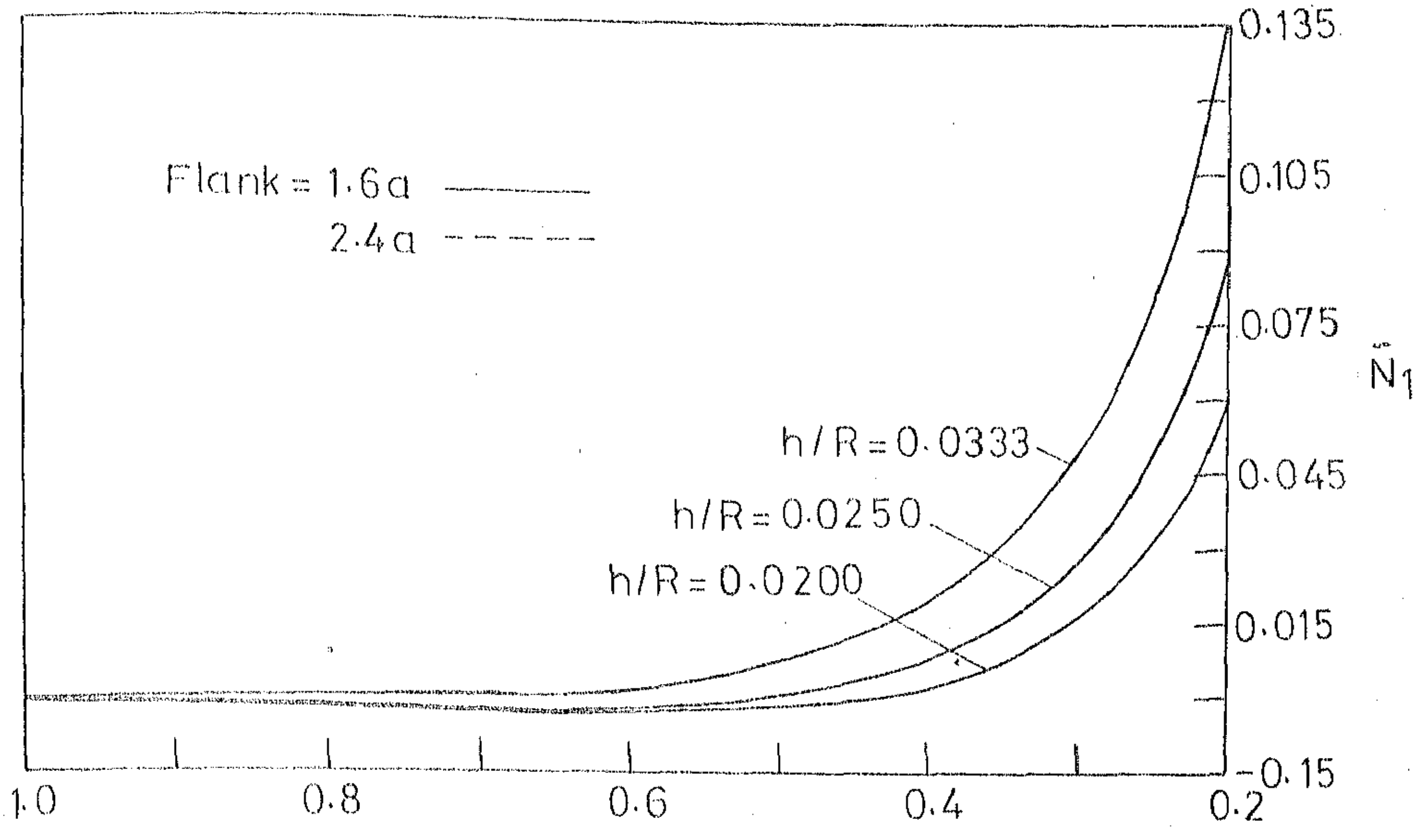


FIG. 3.9c FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{N}_1$ FOR DIFFERENT (h/R) VALUES

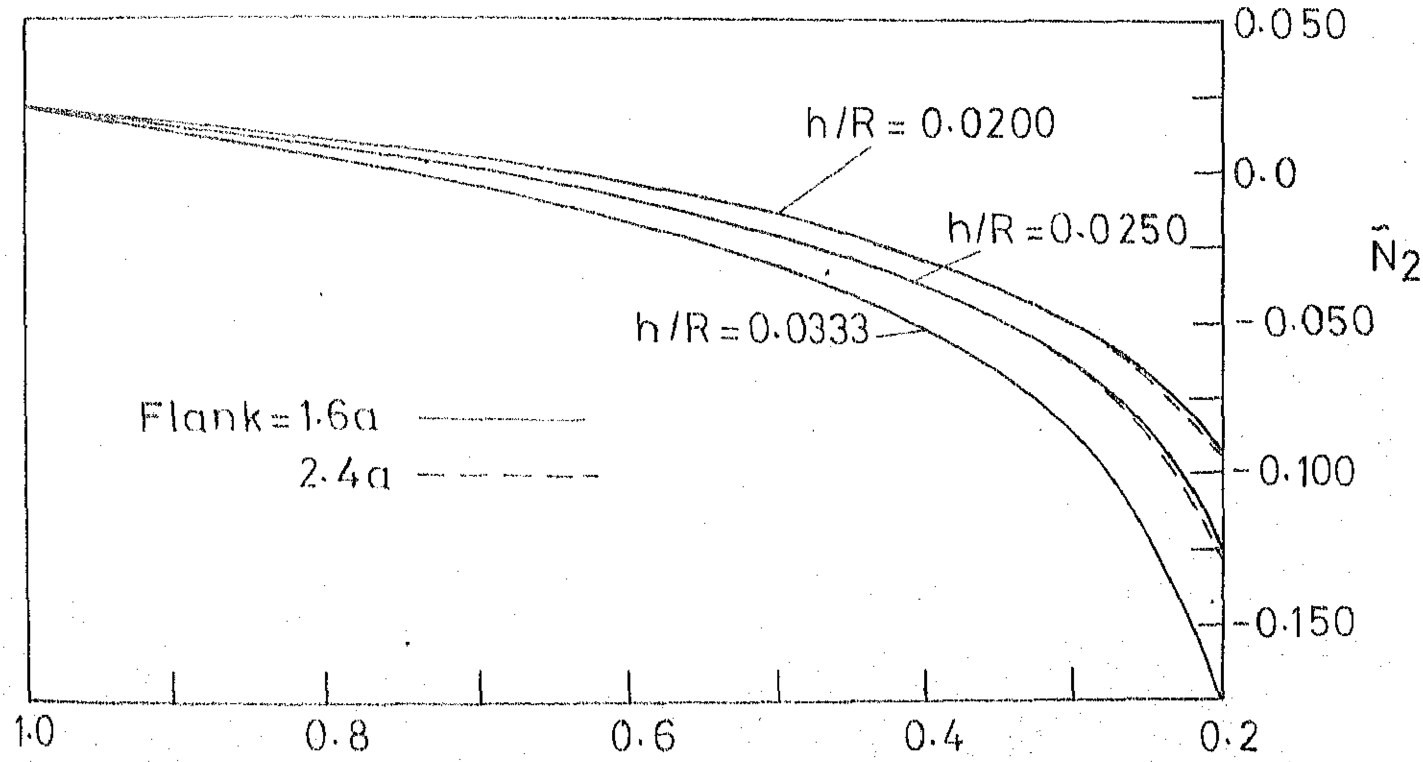


FIG. 3.9d FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{N}_2$ FOR DIFFERENT (h/R) VALUES

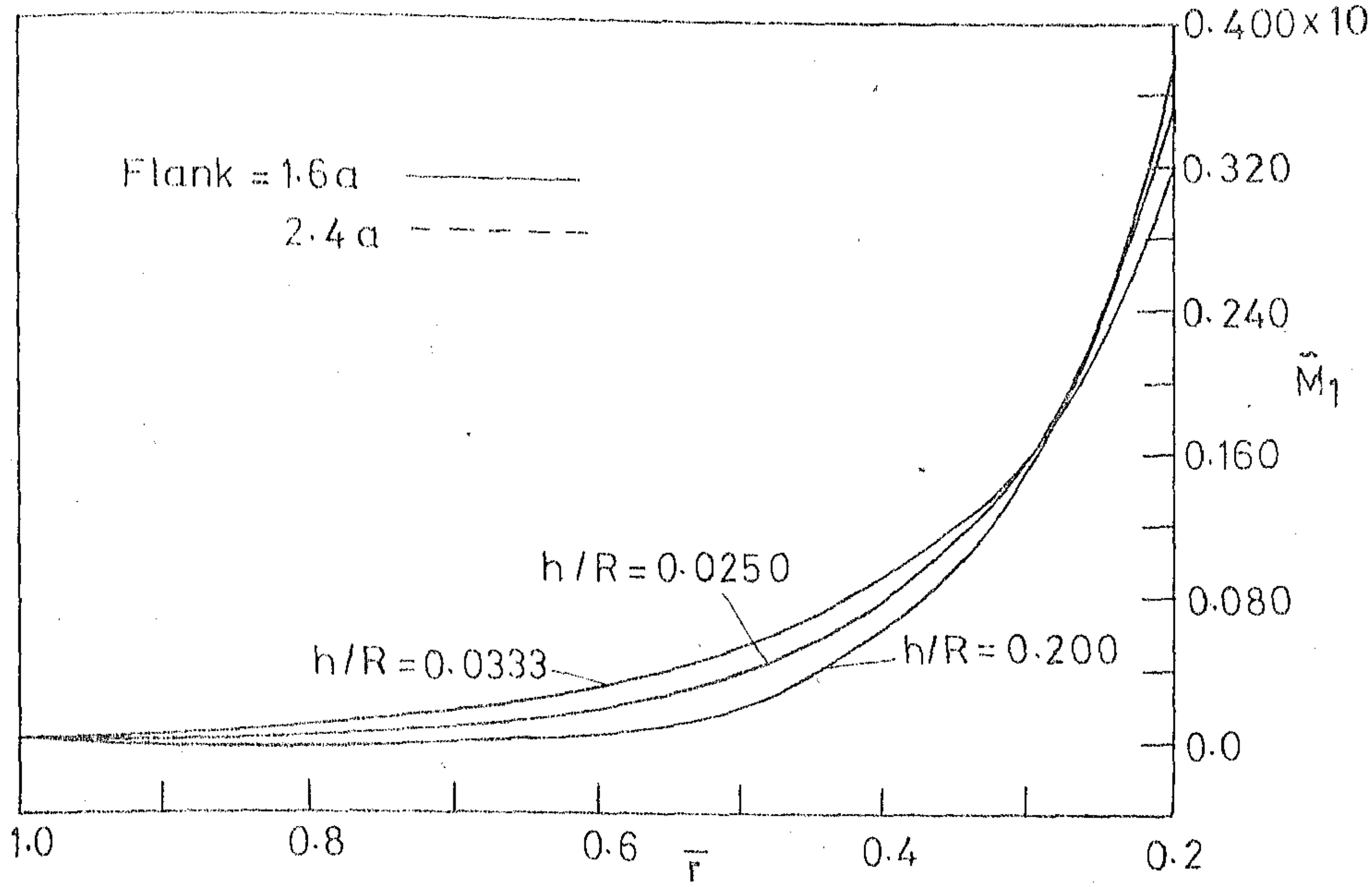


FIG. 3.9e FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_1$ FOR DIFFERENT (h/R) VALUES

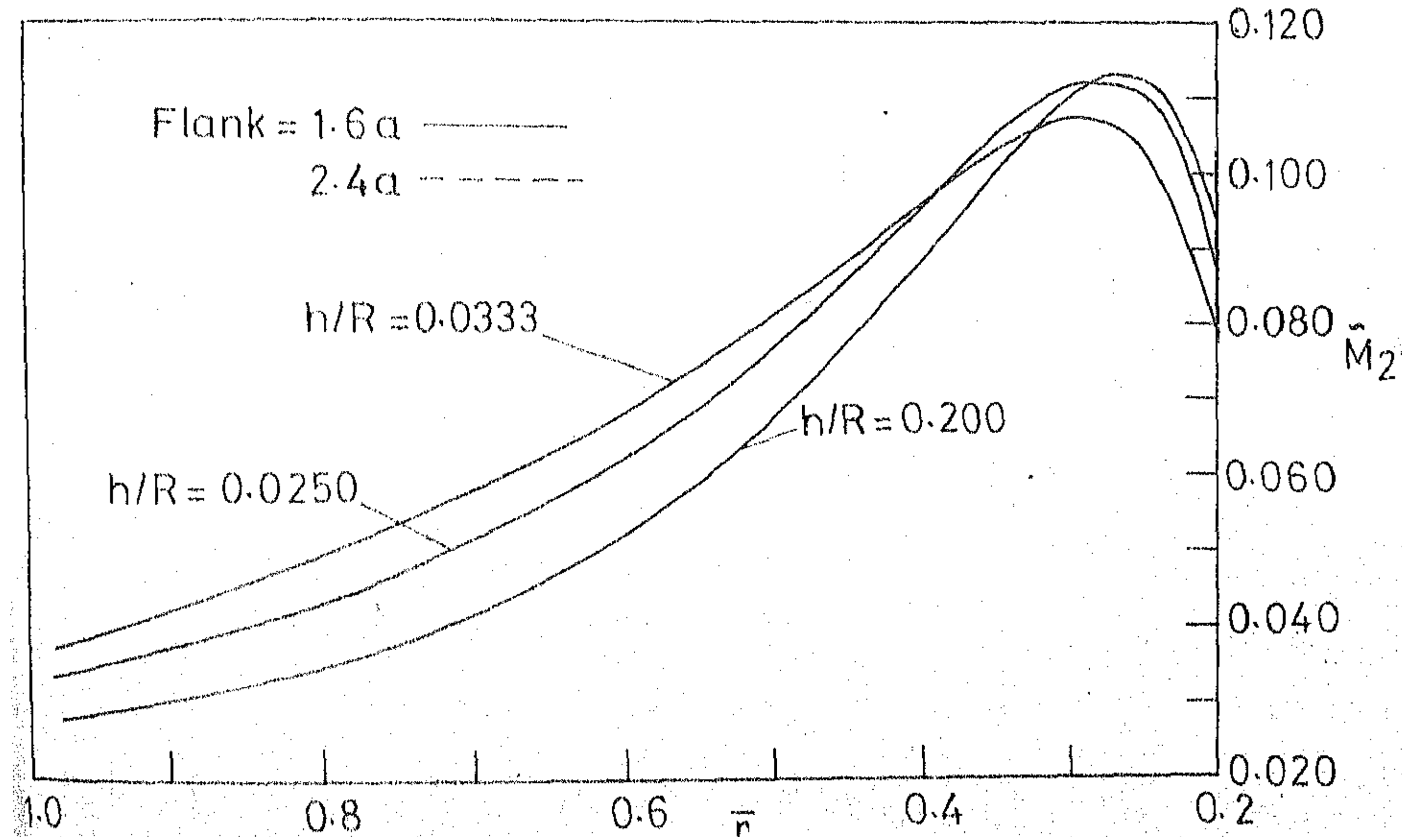


FIG. 3.9f FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_2$ FOR DIFFERENT (h/R) VALUES

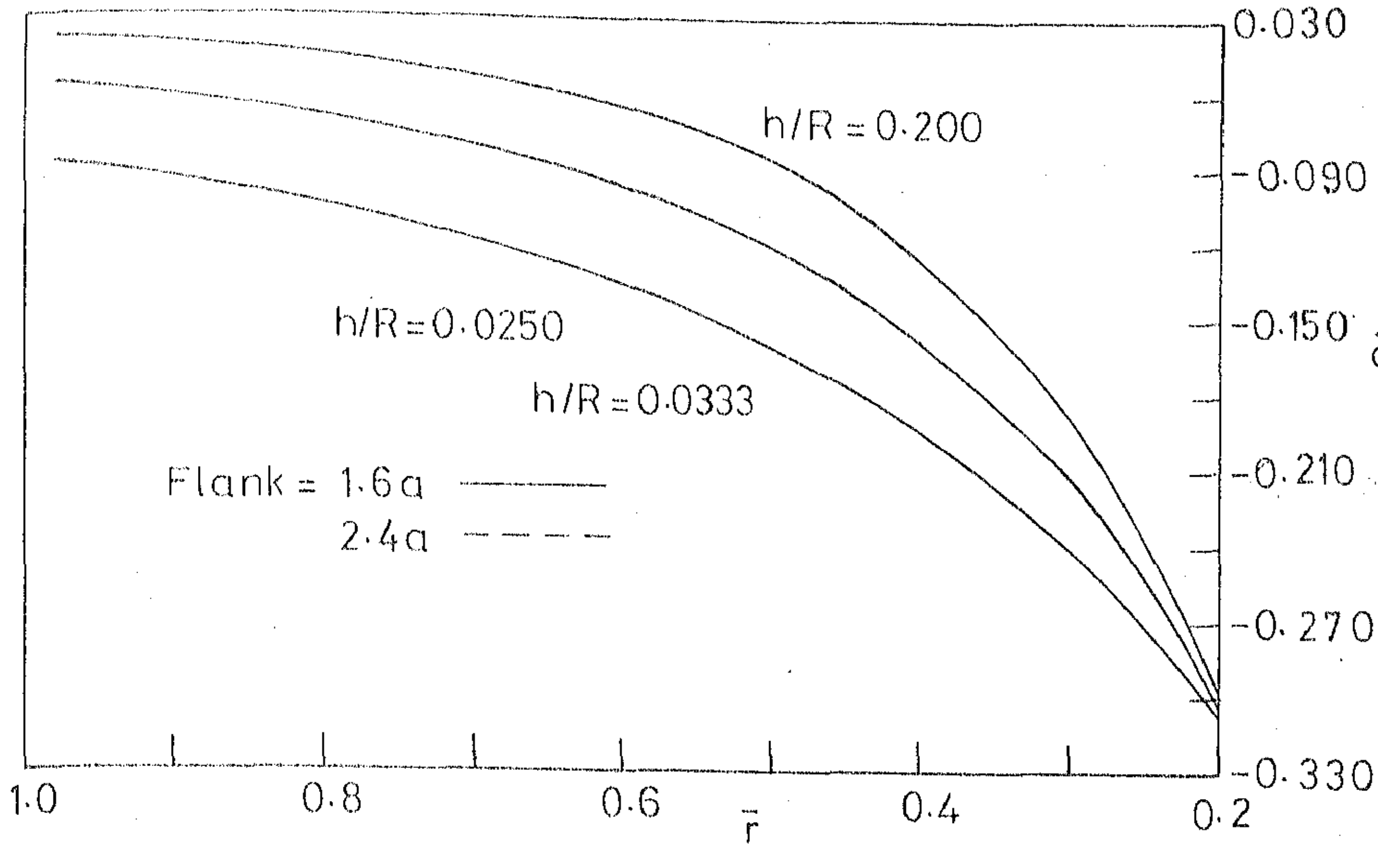


FIG. 3.9q FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (h/R) VALUES

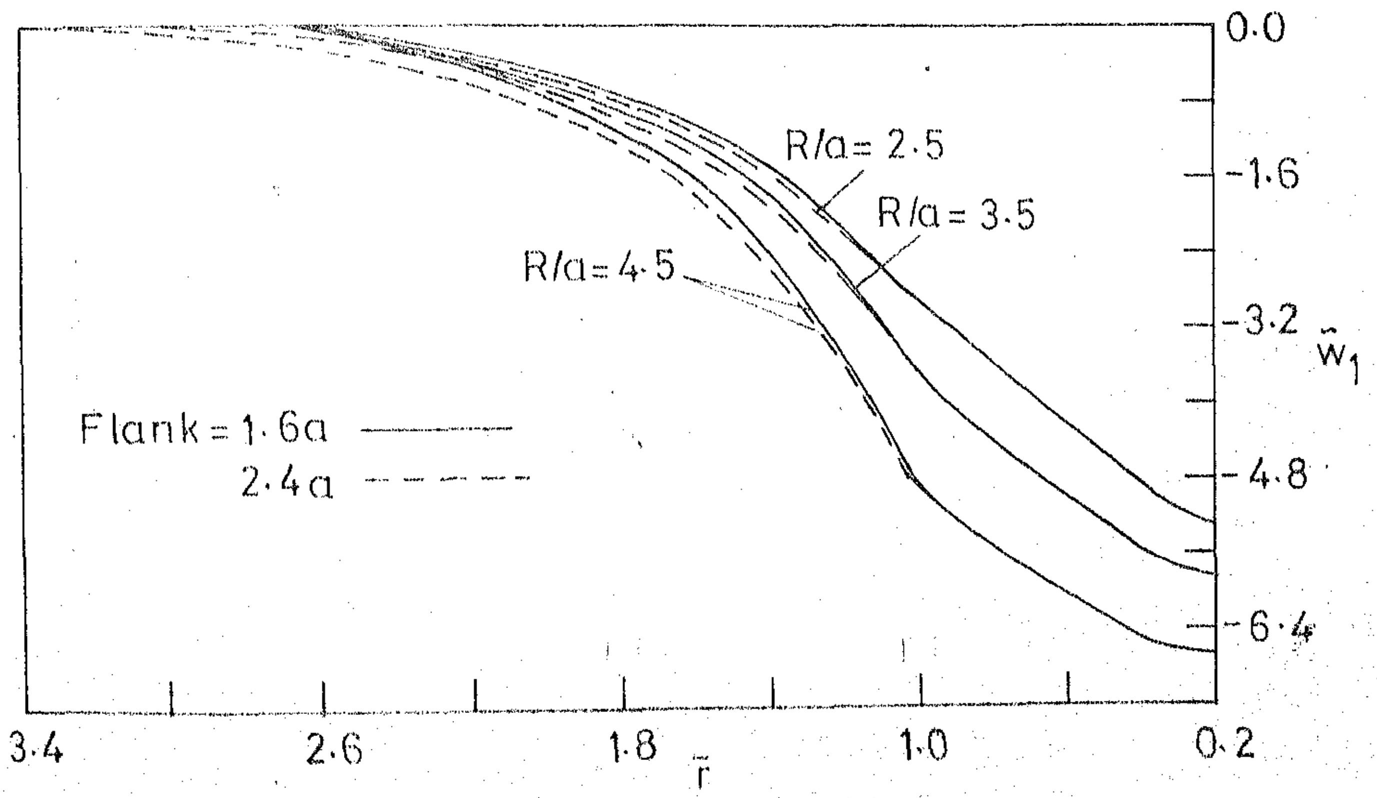


FIG. 3.10a FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (R/a) VALUES

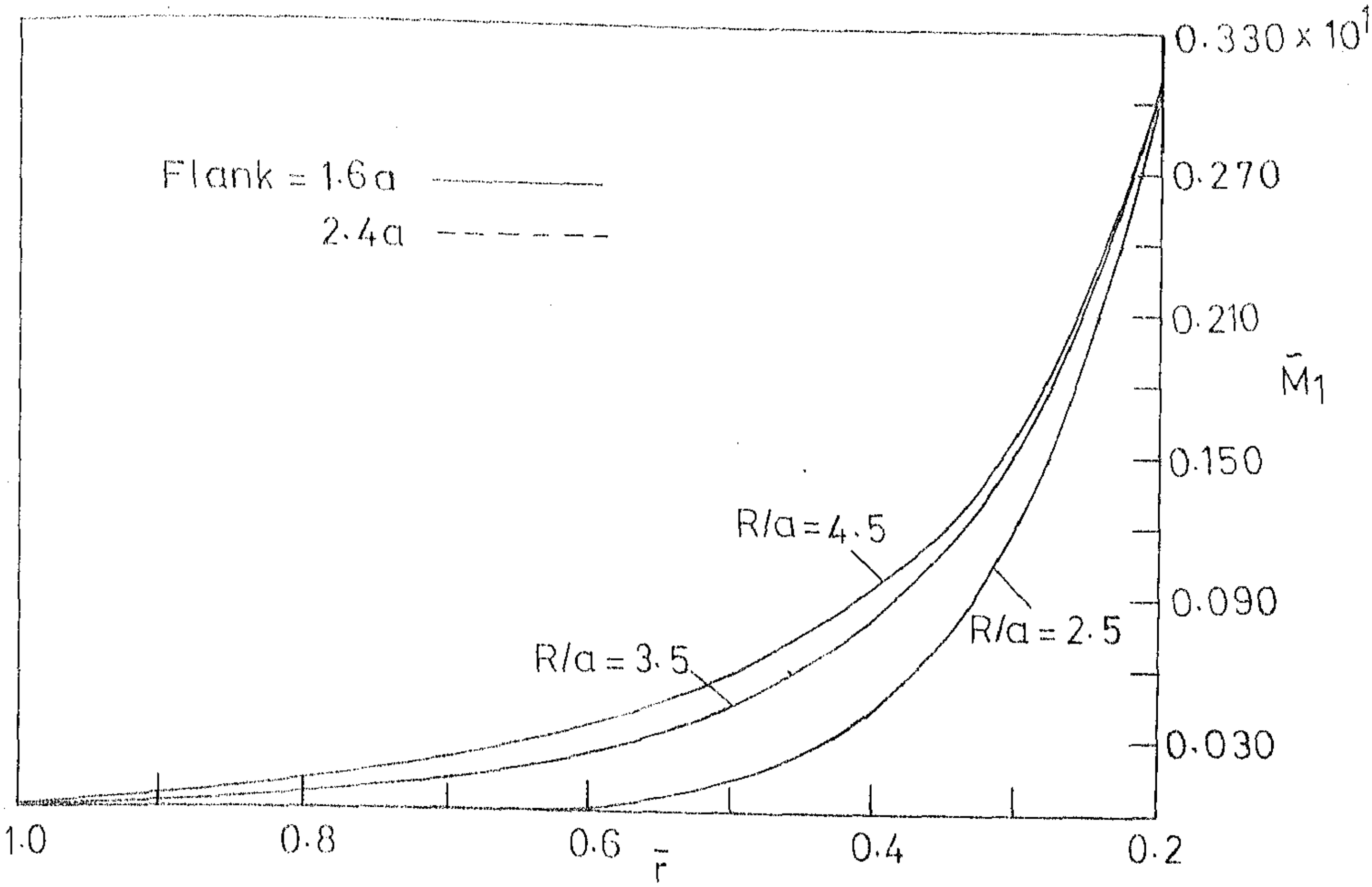


FIG. 3.10b FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{M}_1$ FOR DIFFERENT (R/a) VALUES

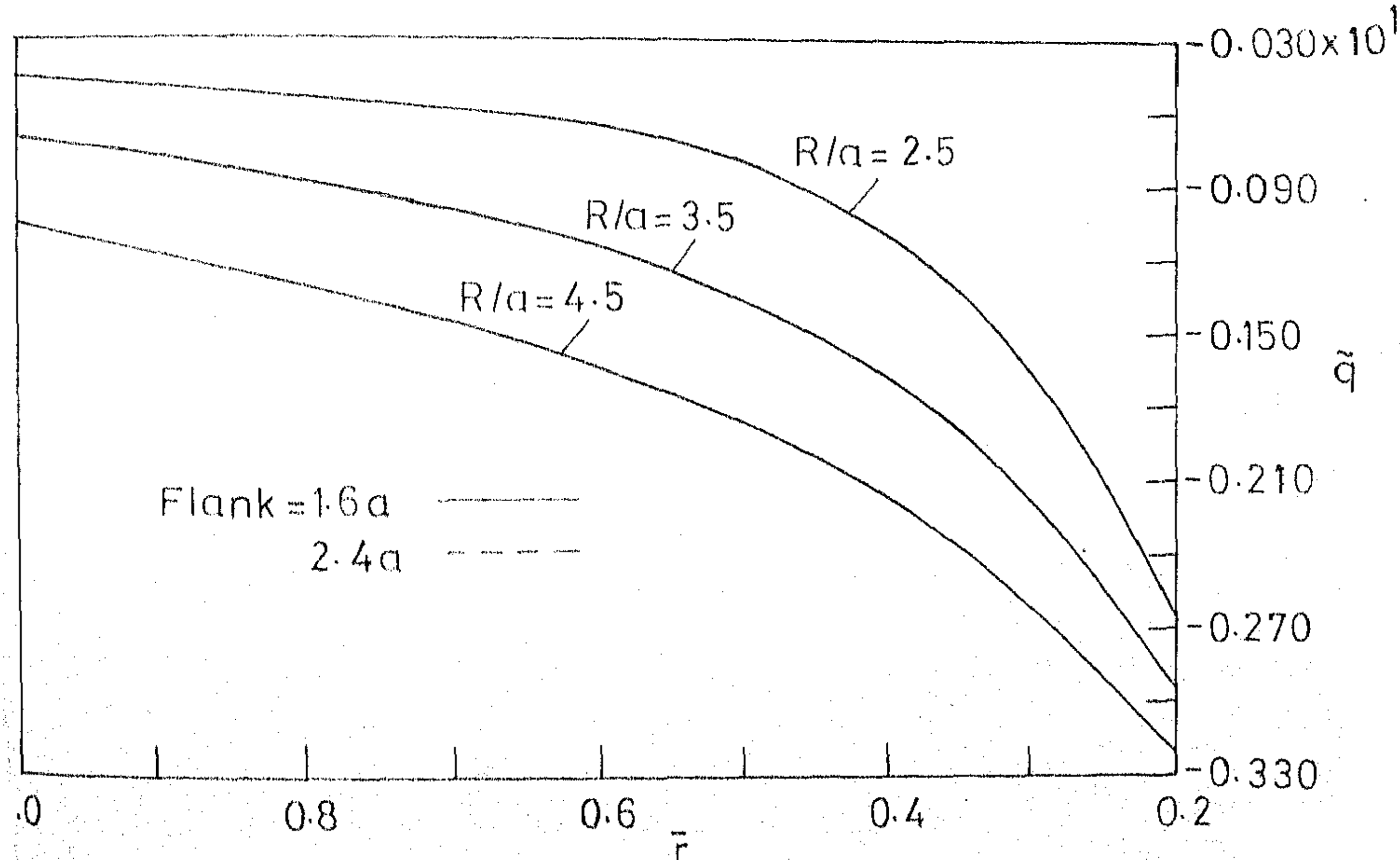


FIG. 3.10c FREE SPHERICAL SHELL; NORMAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (R/a) VALUES

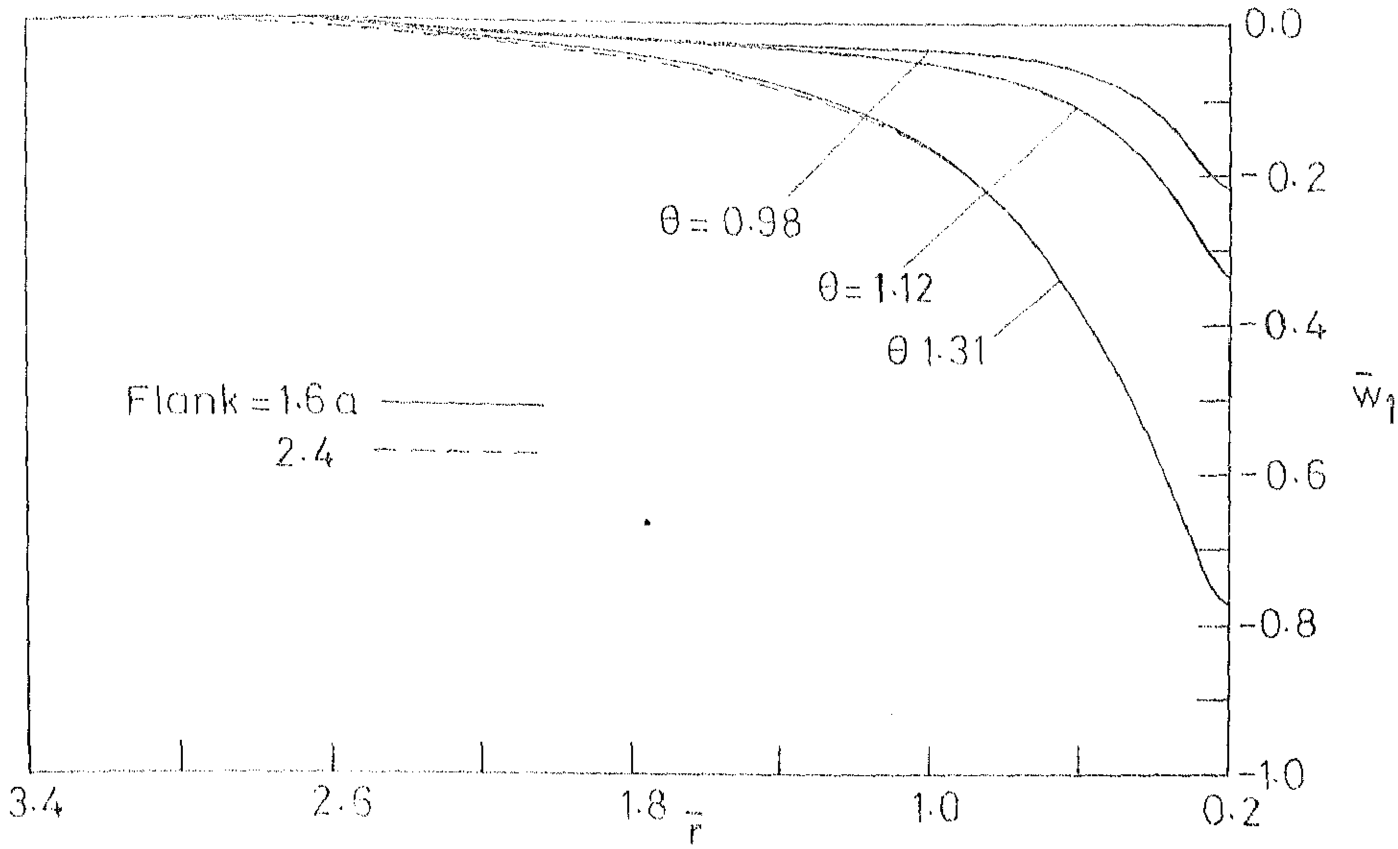


FIG. 3.11a FREE CONICAL SHELL; NORMAL LOAD
VARIATION OF $\bar{w}_1$ FOR DIFFERENT θ VALUES

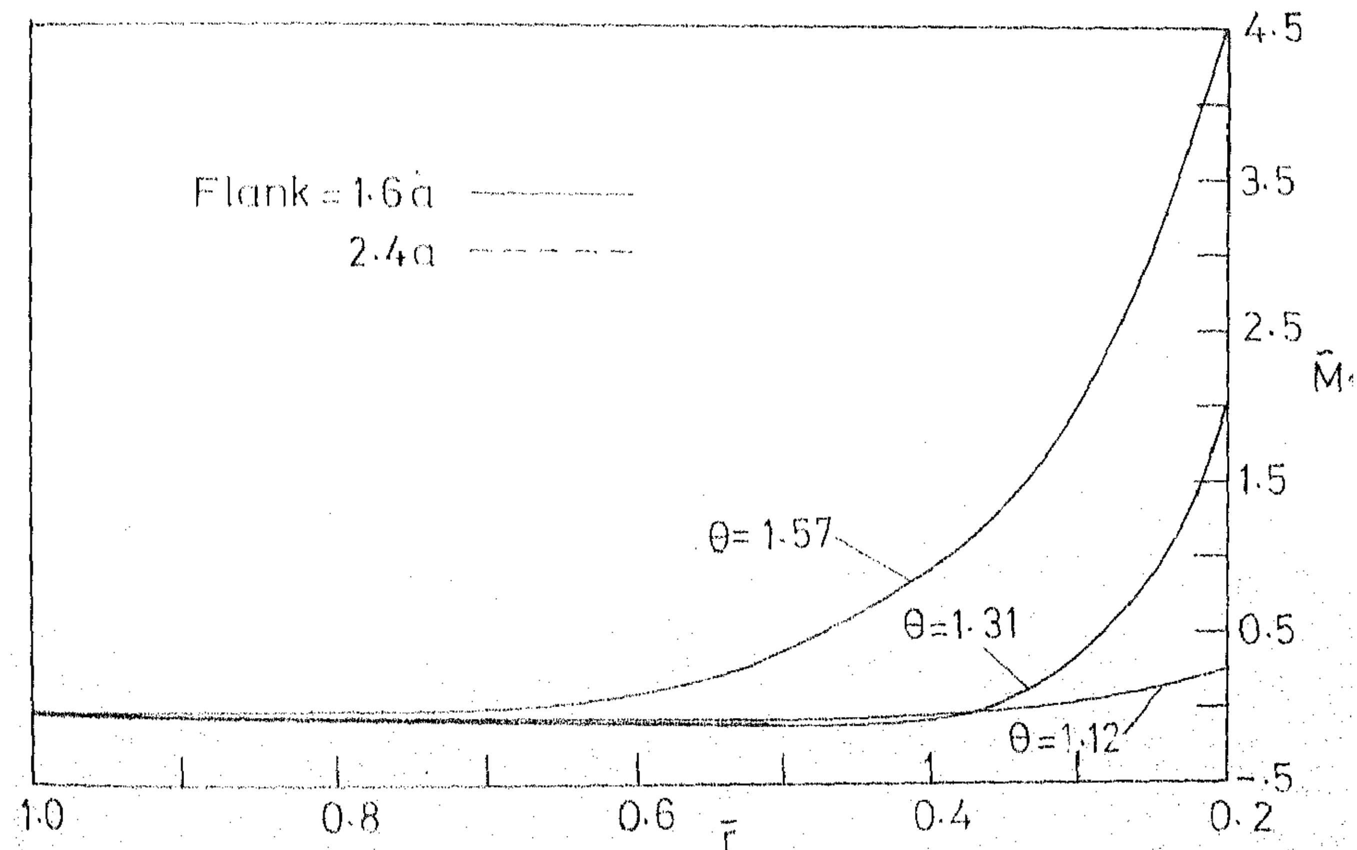


FIG. 3.11b FREE CONICAL SHELL; NORMAL LOAD
VARIATION OF $\bar{M}_1$ FOR DIFFERENT θ VALUES

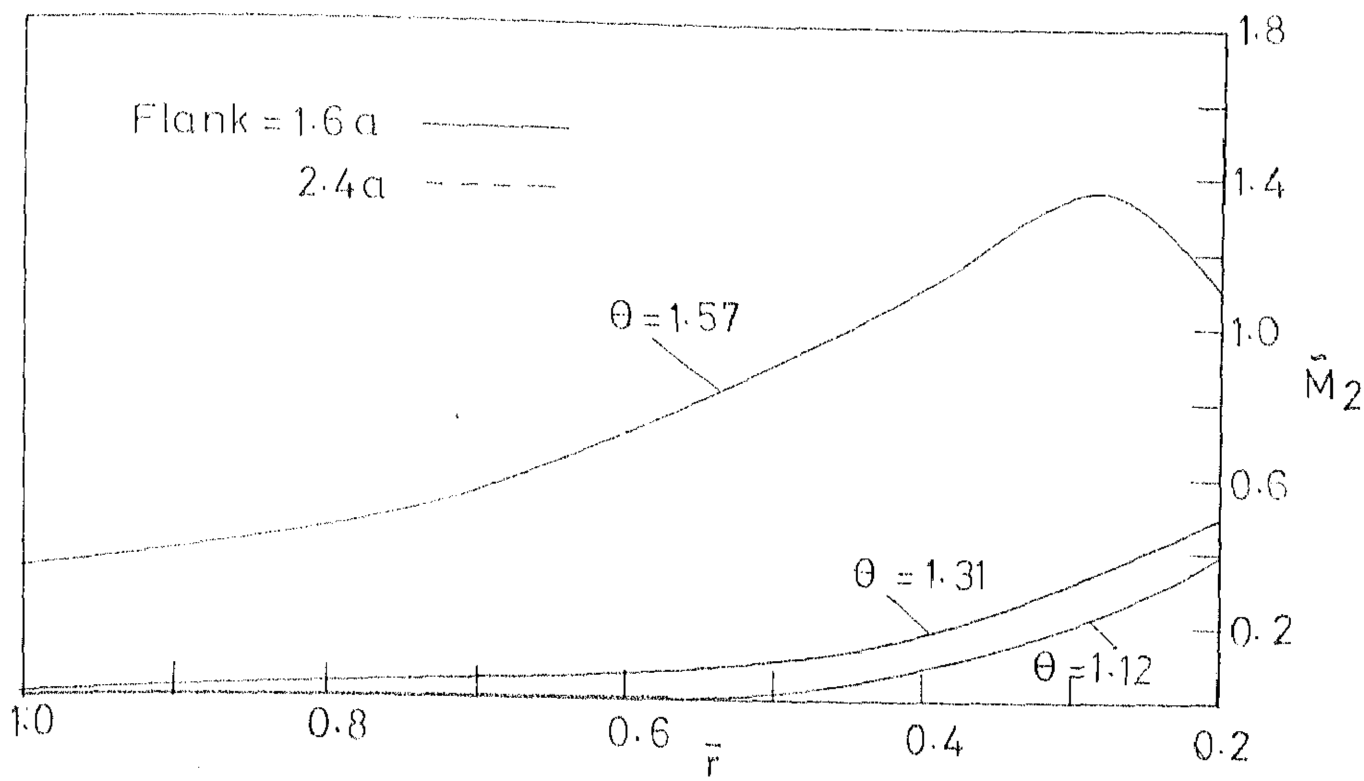


FIG. 3.11c FREE CONICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{M}_2$ FOR DIFFERENT θ VALUES

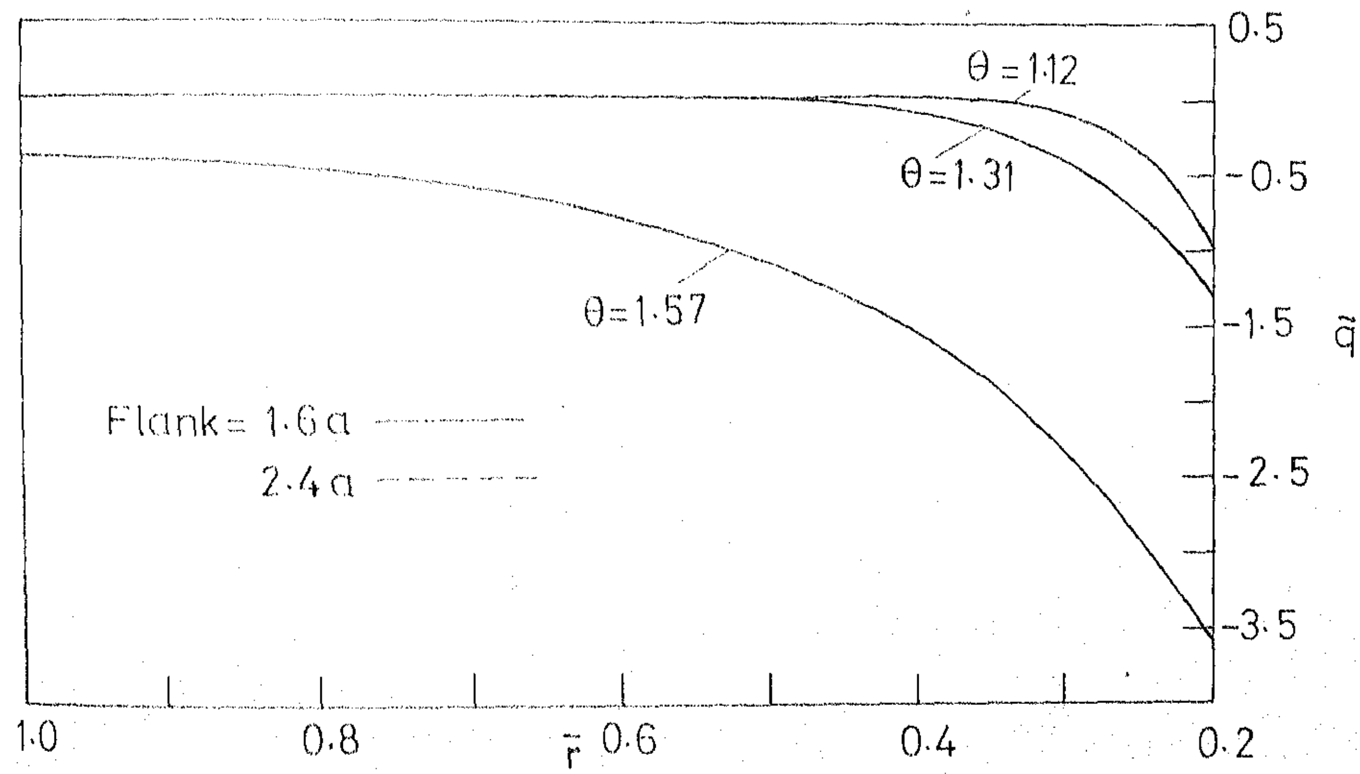


FIG. 3.11d FREE CONICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{q}$ FOR DIFFERENT θ VALAUE

CHAPTER IV

ANTISYMMETRIC PROBLEMS

4.1 GENERAL:

Equations for foundation model and shells developed in Chapter II have been specialised for thin shallow spherical shells and conical shells on elastic foundations, subjected to antisymmetric loads.

Shallow spherical and conical shells subjected to loads in radial direction and moments in radial direction, transferred by the column have been solved for fixed, simply supported and free boundary conditions. The foundation thickness has been considered as finite in all the numerical examples that have been considered. Although, the problems with infinite foundation thicknesses, layered elastic foundations could also be considered without any difficulty. Results have been presented in non-dimensional form so that they can be used for design purposes.

In case of spherical shells two parameters which accounts for thinness of the shell (h/R) and shallowness of the shell (R/a) have been considered. Results have been presented for different values of these parameters. In case of conical shells results are presented for different

values of θ which accounts for the shallowness of the conical shells.

4.2 EQUATIONS FOR THIN SHALLOW SHELLS FOR ASYMMETRIC PROBLEMS:

4.2.1 Spherical Shells:

In asymmetric problems, the displacement functions are dependent on both r and β.

$$u_s = u_s (r, \beta) \tag{4.1a}$$

$$v_s = v_s (r, \beta) \tag{4.1b}$$

and

$$w_s = w_s (r, \beta) \tag{4.1c}$$

In view of the Eqns. (3.1), (3.2), (3.3) and (3.4), which are valid for asymmetric case also, the shell equations in curvilinear coordinates as given in Eqns. (2.25), yields,

$$\frac{\partial^2 u_s}{\partial r^2} + \frac{1}{r}\frac{\partial u_s}{\partial r} - \frac{1}{r^2}\frac{\partial u_s}{\partial r} + \frac{1}{r}\frac{\partial^2 v_s}{\partial r \partial \beta} - \frac{1}{r^2}\frac{\partial v_s}{\partial \beta} + \frac{2}{R}\frac{\partial w_s}{\partial r}$$

$$- (1 - \mu_s) \left[\frac{1}{2r^2}\frac{\partial v_s}{\partial \beta} + \frac{1}{2r}\frac{\partial^2 v_s}{\partial r \partial \beta} - \frac{1}{2r^2}\frac{\partial^2 u_s}{\partial \beta^2}\right]$$

$$+ (1 - \mu_s) \left[\frac{u_s}{R^2} - \frac{1}{R}\frac{\partial w_s}{\partial r}\right] = - \frac{(1-\mu_s{}^2)}{E_s h} X \tag{4.2a}$$

$$\left[\frac{1}{r^2}\frac{\partial u_s}{\partial \beta} + \frac{1}{r}\frac{\partial^2 u_s}{\partial r \partial \beta} + \frac{1}{r^2}\frac{\partial^2 v_s}{\partial \beta^2} + \frac{2}{rR}\frac{\partial w_s}{\partial \beta}\right] + (1 - \mu_s)$$

$$\times \left[\frac{1}{2r}\frac{\partial v_s}{\partial r} - \frac{1}{r^2} v_s + \frac{1}{2}\frac{\partial^2 v_s}{\partial r^2} - \frac{1}{2r}\frac{\partial^2 u_s}{\partial r \partial \beta} + \frac{1}{2r^2}\frac{\partial u_s}{\partial \beta}\right]$$

$$+ (1 - \mu_s) \left[\frac{v_s}{R^2} - \frac{1}{rR}\frac{\partial w_s}{\partial \beta}\right] = - \frac{(1-\mu_s{}^2)}{E_s\, h} Y \tag{4.2b}$$

$$- \left[\frac{2}{R}\frac{u_s}{r} + \frac{2}{R}\frac{\partial u_s}{\partial r} + \frac{2}{rR}\frac{\partial v_s}{\partial \beta} + \frac{4w_s}{R^2} \right] + (1-\mu_s)\left[\frac{2w_s}{R^2}\right.$$

$$\left. + \frac{u_s}{rR} + \frac{1}{R}\frac{\partial u}{\partial r} + \frac{1}{rR}\frac{\partial v_s}{\partial \beta} \right] - \frac{h^2}{12}\left[\frac{2}{rR^2}\frac{\partial w_s}{\partial r}\right.$$

$$\left. + \frac{2}{R^2}\frac{\partial^2 w_s}{\partial r^2} + \frac{2}{r^2R^2}\frac{\partial^2 w_s}{\partial \beta^2} \right] - \frac{h^2}{12}\nabla^2\nabla^2 w_s$$

$$= -\frac{(1-\mu_s^2)}{E_s\, h}\, Z \qquad (4.2c)$$

$$\text{where } \nabla^2\nabla^2 = \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{4}{r^4}\frac{\partial^2}{\partial \beta^2}$$

$$- \frac{2}{r^3}\frac{\partial^3}{\partial r\,\partial \beta^2} + \frac{2}{r^2}\frac{\partial^4}{\partial r^2\,\partial \beta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial \beta^4} \qquad (4.3)$$

Eqns. (4.2a), (4.2b), and (4.2c) represent the equations of equilibrium in terms of displacement functions u_s, v_s and w_s for thin shallow spherical shell and are applicable to asymmetric problems.

4.2.2 Conical Shells:

Proceeding on similar lines, as in case of spherical shells and using Eqns. (3.1), (3.2), (3.7), (3.8) and (4.1), the equations of equilibrium, in terms of displacement functions, for thin shallow conical shell can be written as,

$$\frac{\partial^2 u_s}{\partial r^2} + \frac{1}{r}\frac{\partial u_s}{\partial r} - \frac{u_s}{r^2} - \frac{1}{r^2}\frac{\partial v_s}{\partial \beta} + \frac{1}{r}\frac{\partial^2 v_s}{\partial r \partial \beta} + \frac{\cos\theta}{r}\frac{\partial w_s}{\partial r}$$

$$- \frac{\cos\theta}{r^2} w_s - (1 - \mu_s)\left[-\frac{1}{2r^2}\frac{\partial v_s}{\partial \beta} + \frac{1}{2r}\frac{\partial^2 v_s}{\partial r \partial \beta}\right.$$

$$\left. - \frac{1}{2r^2}\frac{\partial^2 u_s}{\partial \beta^2} + \frac{\cos\theta}{r}\frac{\partial w_s}{\partial r}\right] = -\frac{(1-\mu_s^2)}{E_s h} X \quad (4.4a)$$

$$\frac{1}{r^2}\frac{\partial u_s}{\partial \beta} + \frac{1}{r}\frac{\partial^2 u_s}{\partial r \partial \beta} + \frac{1}{r^2}\frac{\partial^2 v_s}{\partial \beta^2} + \frac{\cos\theta}{r^2}\frac{\partial w_s}{\partial \beta} + \frac{(1-\mu_s)}{2}\left[\frac{\partial^2 v_s}{\partial r^2}\right.$$

$$\left. + \frac{1}{r}\frac{\partial v_s}{\partial r} - \frac{1}{r^2} v_s + \frac{1}{r^2}\frac{\partial u_s}{\partial \beta} - \frac{1}{r}\frac{\partial^2 u_s}{\partial r \partial \beta}\right] = -\frac{(1-\mu_s^2)}{E_s h} Y$$

$$(4.4b)$$

$$-\frac{\cos\theta}{r}\left[\frac{u_s}{r} + \frac{\partial u_s}{\partial r} + \frac{1}{r}\frac{\partial v_s}{\partial \beta} + \frac{\cos\theta}{r} w_s - (1-\mu_s)\frac{\partial u_s}{\partial r}\right]$$

$$- \frac{h^2}{12}\frac{\cos^2\theta}{r^2}\left[\frac{1}{r^2}\frac{\partial^2 w_s}{\partial \beta^2} + \frac{\partial^2 w_s}{\partial r^2} - \frac{3}{r}\frac{\partial w_s}{\partial r}\right]$$

$$- \frac{h^3}{3}\frac{\cos^2\theta}{r^4} w_s - \frac{h^2}{12}\nabla^2\nabla^2 w_s = -\frac{(1 - \mu_s^2)}{E_s h} Z$$

$$(4.4c)$$

where $\nabla^2\nabla^2$ is given by Eqn. (4.3).

4.3 EQUATIONS FOR ELASTIC FOUNDATIONS APPLICABLE TO ASYMMETRIC PROBLEMS:

Equations for elastic foundation developed in Chapter II are specialised to suit the foundations for shallow spherical shells and shallow conical shells.

In view of the Eqns. (3.11), (3.12), the foundation equations in curvilinear coordinates given by Eqns. (2.13) yield,

$$\sum_{i=1}^{l} a_{if} \frac{\partial^2 u_i}{\partial r^2} + \sum_{i=1}^{l} \frac{a_{if}}{r} \frac{\partial u_i}{\partial r} - \sum_{i=1}^{l} \frac{a_{if}}{r^2} u_i - \sum_{i=1}^{l} b_{if} u_i$$

$$+ \sum_{i=1}^{l} \frac{h_{if}}{r^2} \frac{\partial^2 u_i}{\partial \beta^2} - \sum_{j=1}^{m} \frac{f_{jf}}{r^2} \frac{\partial v_j}{\partial \beta} + \sum_{j=1}^{m} \frac{t_{jf}}{r} \frac{\partial^2 v_j}{\partial r \partial \beta}$$

$$+ \sum_{k=1}^{n} d_{kf} \frac{\partial w_k}{\partial r} - \sum_{k=1}^{n} c_{kf} \frac{\partial w_k}{\partial r} = - \overline{P}_{rf} \qquad (4.5a)$$

$$(f = 1, 2, \ldots \quad)$$

$$\sum_{j=1}^{m} m_{jg} \frac{\partial^2 v_j}{\partial r^2} + \sum_{j=1}^{m} \frac{m_{jg}}{r} \frac{\partial v_j}{\partial r} - \sum_{j=1}^{m} \frac{m_{jg}}{r^2} v_j - \sum_{j=1}^{m} n_{jg} v_j$$

$$+ \sum_{j=1}^{m} \frac{e_{jg}}{r^2} \frac{\partial^2 v_j}{\partial \beta^2} + \sum_{i=1}^{l} \frac{t_{ig}}{r} \frac{\partial^2 u_i}{\partial r \partial \beta} + \sum_{i=1}^{l} \frac{f_{ig}}{r^2} \frac{\partial u_i}{\partial \beta}$$

$$+ \sum_{k=1}^{n} \frac{l_{kg}}{r} \frac{\partial w_k}{\partial \beta} - \sum_{k=1}^{n} \frac{k_{kg}}{r} \frac{\partial w_k}{\partial \beta} = - \overline{P}_{\beta g} \qquad (4.5b)$$

$$\sum_{k=1}^{n} r_{kh} \frac{\partial^2 w_k}{\partial r^2} + \sum_{k=1}^{n} \frac{r_{kh}}{r} \frac{\partial w_k}{\partial r} - \sum_{k=1}^{n} s_{kh} w_k$$

$$+ \sum_{k=1}^{n} \frac{r_{kh}}{r^2} \frac{\partial^2 w_k}{\partial \beta^2} - \sum_{i=1}^{l} d_{ih} \frac{\partial u_i}{\partial r} + \sum_{i=1}^{l} c_{ih} \frac{\partial u_i}{\partial r}$$

$$- \sum_{i=1}^{l} \frac{d_{ih}}{r} u_i + \sum_{i=1}^{l} \frac{c_{ih}}{r} u_i - \sum_{j=1}^{m} \frac{l_{jh}}{r} \frac{\partial v_j}{\partial \beta}$$

$$+ \sum_{j=1}^{m} \frac{k_{jh}}{r} \frac{\partial v_j}{\partial \beta} = - \overline{P}_{rh} \qquad (4.5c)$$

where,

$$a_{if} = \int_0^H \frac{E_o}{(1 - \mu_o^2)} \phi_i \phi_f \, d\gamma \tag{4.6a}$$

$$b_{if} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \phi_i' \phi_f' \, d\gamma \tag{4.6b}$$

$$c_{kf} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \psi_k \phi_f' \, d\gamma \tag{4.6c}$$

$$c_{ih} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \phi_i' \psi_h \, d\gamma \tag{4.6d}$$

$$d_{kf} = \int_0^H \frac{E_o \mu_o}{(1 - \mu_o^2)} \psi_k' \phi_f \, d\gamma \tag{4.6e}$$

$$d_{ih} = \int_0^H \frac{E_o \mu_o}{(1 - \mu_o^2)} \phi_i \psi_h' \, d\gamma \tag{4.6f}$$

$$f_{jf} = \int_0^H \frac{(3 - \mu_o) E_o}{2(1 - \mu_o^2)} \chi_j \phi_f \, d\gamma \tag{4.6g}$$

$$f_{ig} = \int_0^H \frac{(3 - \mu_o) E_o}{2(1 - \mu_o^2)} \phi_i \chi_g \, d\gamma \tag{4.6h}$$

$$g_{jg} = \int_0^H \frac{E_o}{(1 - \mu_o^2)} \chi_j \chi_g \, d\gamma \tag{4.6i}$$

$$h_{if} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \phi_i \phi_f \, d\gamma \tag{4.6j}$$

$$k_{jh} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \chi_j' \psi_h \, d\gamma \tag{4.6k}$$

$$k_{kg} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \psi_k \chi_g' \, d\gamma \tag{4.6l}$$

$$l_{jh} = \int_0^H \frac{E_o \mu_o}{(1 - \mu_o^2)} \chi_j \psi'_h \, d\gamma \qquad (4.6m)$$

$$l_{kg} = \int_0^H \frac{E_o \mu_o}{(1 - \mu_o^2)} \psi'_k \chi_g \, d\gamma \qquad (4.6n)$$

$$m_{jg} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \chi_j \chi_g \, d\gamma \qquad (4.6p)$$

$$n_{jg} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \chi'_j \chi'_g \, d\gamma \qquad (4.6q)$$

$$r_{kh} = \int_0^H \frac{E_o}{2(1 + \mu_o)} \psi_k \psi_h \, d\gamma \qquad (4.6r)$$

$$s_{kh} = \int_0^H \frac{E_o}{(1 - \mu_o^2)} \psi'_k \psi'_h \, d\gamma \qquad (4.6s)$$

$$t_{ig} = \int_0^H \frac{E_o}{2(1 - \mu_o)} \phi_i \chi_g \, d\gamma \qquad (4.6t)$$

$$t_{jf} = \int_0^H \frac{E_o}{2(1 - \mu_o)} \chi_j \phi_f \, d\gamma \qquad (4.6u)$$

$$E_o = \frac{E}{(1 - \mu^2)} \qquad (4.6v)$$

$$\mu_o = \frac{\mu}{1 - \mu} \qquad (4.6w)$$

$$\overline{P}_{rf} = \int_0^H P_r \, (r, \beta, \gamma) \, \phi_f \, d\gamma + \left[\sigma_{r\gamma} \, \phi_f \, (\gamma)\right]_0^H \qquad (4.6x)$$

$$\overline{P}_{\beta g} = \int_0^H P_\beta \, (r, \beta, \gamma) \, \chi_g \, d\gamma + \left[\sigma_{\beta\gamma} \, \chi_g \, (\gamma)\right]_0^H \qquad (4.6y)$$

$$\overline{P}_{\gamma h} = \int_0^H P_\gamma \, (r, \beta, \gamma) \, \psi_h \, d\gamma + \left[\sigma_{\gamma\gamma} \psi_h \, (\gamma)\right]_0^H \qquad (4.6z)$$

where H is the thickness of the foundation and prime denotes derivatives with respect to γ, E is the modulus of elasticity of the foundation and μ is the Poisson's ratio. Eqns. (4.5) represents the elastic foundation for asymmetric problems. The elastic foundation is characterised by the constants which depend on geometric and elastic properties of the body [Eqns. (4.6)]. In this model, elastic foundations of both finite and infinite thicknesses and layered elastic foundations can be considered.

Taking only one term of the finite series [Eqns. (2.4)], the displacement functions can be written as,

$$\begin{aligned} u\ (r,\ \beta,\ \gamma) &= u_1\ (r,\ \beta)\ \phi_1\ (\gamma) \\ v\ (r,\ \beta,\ \gamma) &= v_1\ (r,\ \beta)\ \phi_1\ (\gamma) \qquad (4.7) \\ w\ (r,\ \beta,\ \gamma) &= w_1\ (r,\ \beta)\ \psi_1\ (\gamma) \end{aligned}$$

Substituting Eqns. (4.6) in Eqns. (4.5), the resulting expressions are,

$$a_{11} \frac{\partial^2 u_1}{\partial r^2} + \frac{a_{11}}{r} \frac{\partial u_1}{\partial r} - \frac{a_{11}}{r^2} u_1 - b_{11} u_1 + \frac{h_{11}}{r^2} \frac{\partial^2 u_1}{\partial \beta^2}$$

$$- \frac{f_{11}}{r^2} \frac{\partial v_1}{\partial \beta} + \frac{t_{11}}{r} \frac{\partial^2 v_1}{\partial \beta \partial r} + d_{11} \frac{\partial w_1}{\partial r}$$

$$- e_{11} \frac{\partial w_1}{\partial r} = - \bar{P}_{r1} \qquad (4.8a)$$

$$m_{11}\frac{\partial^2 v_1}{\partial r^2}+\frac{m_{11}}{r}\frac{\partial v_1}{\partial r}-\frac{m_{11}}{r^2}v_1-n_{11}\,v_1+\frac{g_{11}}{r^2}\frac{\partial^2 v_1}{\partial \beta^2}$$

$$+\frac{t_{11}}{r}\frac{\partial^2 u_1}{\partial r\partial \beta}+\frac{f_{11}}{r^2}\frac{\partial u_1}{\partial \beta}+\frac{l_{11}}{r}\frac{\partial w_1}{\partial \beta}-\frac{k_{11}}{r}\frac{\partial w_1}{\partial \beta}$$

$$=-\bar{P}_{\beta 1} \tag{4.8b}$$

$$r_{11}\frac{\partial^2 w_1}{\partial r^2}+\frac{r_{11}}{r^2}\frac{\partial^2 w_1}{\partial \beta^2}+\frac{r_{11}}{r}\frac{\partial w_1}{\partial r}-s_{11}w_1-d_{11}\frac{\partial u_1}{\partial r}$$

$$+c_{11}\frac{\partial u_1}{\partial r}-\frac{d_{11}}{r}u_1+\frac{c_{11}}{r}u_1-\frac{l_{11}}{r}\frac{\partial v_1}{\partial \beta}$$

$$+\frac{k_{11}}{r}\frac{\partial v_1}{\partial \beta}=-\bar{P}_{\gamma 1} \tag{4.8c}$$

where,

$$a_{11}=\int_0^H \frac{E_o}{(1-\mu_o^2)}\,\phi_1^2\,d\gamma \tag{4.9a}$$

$$b_{11}=\int_0^H \frac{E_o}{2(1+\mu_o)}\,(\phi_1')^2\,d\gamma \tag{4.9b}$$

$$c_{11}=\int_0^H \frac{E_o}{2(1+\mu_o)}\,\phi_1'\,\psi_1\,d\gamma \tag{4.9c}$$

$$d_{11}=\int_0^H \frac{E_o\,\mu_o}{(1-\mu_o^2)}\,\phi_1\,\psi_1'\,d\gamma \tag{4.9d}$$

$$f_{11}=\int_0^H \frac{(3-\mu_o)}{2(1-\mu_o^2)}\,\phi_1\,\chi_1\,d\gamma \tag{4.9e}$$

$$g_{11}=\int_0^H \frac{E_o}{(1-\mu_o^2)}\,\chi_1^2\,d\gamma \tag{4.9f}$$

$$h_{11} = \int_0^H \frac{E_o}{2(1+\mu_o)} \phi_1^2 \, d\gamma \tag{4.9g}$$

$$k_{11} = \int_0^H \frac{E_o}{2(1+\mu_o)} \gamma_1' \psi_1 \, d\gamma \tag{4.9h}$$

$$l_{11} = \int_0^H \frac{E_o \mu_o}{(1-\mu_o^2)} \chi_1' \psi_1' \, d\gamma \tag{4.9i}$$

$$m_{11} = \int_0^H \frac{E_o}{2(1+\mu_o)} \chi_1^2 \, d\gamma \tag{4.9j}$$

$$n_{11} = \int_0^H \frac{E_o}{2(1+\mu_o)} (\chi_1')^2 \, d\gamma \tag{4.9k}$$

$$r_{11} = \int_0^H \frac{E_o}{2(1+\mu_o)} \psi_1^2 \, d\gamma \tag{4.9l}$$

$$s_{11} = \int_0^H \frac{E_o}{(1-\mu_o^2)} (\psi_1')^2 \, d\gamma \tag{4.9m}$$

$$t_{11} = \int_0^H \frac{E_o}{2(1-\mu_o)} \phi_1 \chi_1 \, d\gamma \tag{4.9n}$$

$$\overline{P}_{r1} = \int_0^H P_r(r, \beta, \gamma) \phi_1(\gamma) d\gamma + [\sigma_{r\gamma} \phi_1(\gamma)]_0^H \tag{4.9p}$$

$$\overline{P}_{\beta 1} = \int_0^H P_\beta(r, \beta, \gamma) \chi_1(\gamma) d\gamma + [\sigma_{\beta\gamma} \chi_1(\gamma)]_0^H \tag{4.9q}$$

$$\overline{P}_{\gamma 1} = \int_0^H P_\gamma(r, \beta, \gamma) \psi_1(\gamma) d\gamma + [\sigma_{\gamma\gamma} \psi_1(\gamma)]_0^H \tag{4.9r}$$

In case of externally applied loads on the surface of the body (along $\gamma = 0$), expressions (4.9p) to (4.9r) have to be taken as

Stieltjes integrals, as given below;

$$\bar{P}_{r1} = P_r(r, \beta)\, \phi_1(0) + \int_0^H P_r(\alpha, \beta, \gamma)\, \phi_1(\gamma)\, d\gamma$$

$$+ [\sigma_{r\gamma}\, \phi_1(\gamma)]_0^H \tag{4.10a}$$

$$\bar{P}_{\beta 1} = P_\beta(r, \beta)\, \chi_1(0) + \int_0^H P_\beta(\alpha, \beta, \gamma)\, \chi_1(\gamma)\, d\gamma$$

$$+ [\sigma_{\beta\gamma}\, \chi_1(\gamma)]_0^H \tag{4.10b}$$

$$\bar{P}_{\gamma 1} = P_\gamma(r, \beta)\, \psi_1(0) + \int_0^H P_\gamma(\alpha, \beta, \gamma)\, \psi_1(\gamma)\, d\gamma$$

$$+ [\sigma_{\gamma\gamma}\, \psi_1(\gamma)]_0^H \tag{4.10c}$$

where $P_r(r, \beta)$, $P_\beta(r, \beta)$ and $P_\gamma(r, \beta)$ are the externally applied surface loads in the radial, circumferential and normal directions respectively. The body forces $P_r(\alpha, \beta, \gamma)$, $P_\beta(\alpha, \beta, \gamma)$ and $P_\gamma(\alpha, \beta, \gamma)$ are neglected.

This gives,

$$\bar{P}_{r1} = P_r(r, \beta)\, \phi_1(0) + [\sigma_{r\gamma}\, \phi_1(\gamma)]_0^H \tag{4.11a}$$

$$\bar{P}_{\beta 1} = P_\beta(r, \beta)\, \chi_1(0) + [\sigma_{\beta\gamma}\, \chi_1(\gamma)]_0^H \tag{4.11b}$$

$$\bar{P}_{\gamma 1} = P_\gamma(r, \beta)\, \psi_1(0) + [\sigma_{\gamma\gamma}\, \psi_1(\gamma)]_0^H \tag{4.11c}$$

Since stresses $\sigma_{r\gamma}$, $\sigma_{\beta\gamma}$, $\sigma_{\gamma\gamma}$ are zero at the surface of the foundation and flank and $\gamma = H$ choosing $\phi_1(\gamma) = \lambda_1(\gamma) = \psi_1(\gamma) = 0$, it can be seen that,

$$[\sigma_{r\gamma}\,\phi_1(\gamma)]_0^H = [\sigma_{\beta\gamma}\lambda_1(\gamma)]_0^H = [\sigma_{\gamma\gamma}\psi_1(\gamma)]_0^H = 0 \quad (4.12)$$

4.4 SHELL FOUNDATION INTERACTION EQUATIONS:

4.4.1 Thin Shallow Spherical Shell on Elastic Foundation Subjected to Anti-symmetric Loads:

Consider a thin shallow spherical shell on an elastic foundation carrying asymmetric loads. It is assumed that the friction between the shell and the foundation is negligible. Hence the shell transfers only normal reaction on the supporting medium. Therefore, the external load, on the shell consists of the known forces Z and the foundation reaction P_γ (all referred to unit area).

$$Z_1 = Z - P_\gamma \qquad (4.13)$$

As the shell is transferring only normal forces to the foundation and assuming the compatibility of the normal deflections of the shell and the foundation surface, P_γ can be eliminated using Eqn. (4.8c). Considering that $P_r(r, \beta)$, $P_\beta(r, \beta)$ are equal to zero in Eqns. (4.8a) and (4.8b) and keeping in view the Eqns. (4.12), the resulting expressions for the displacements of the shell foundation system can be written as [Choosing $\psi_1(0) = 1$],

$$a_{11} \frac{\partial^2 u_1}{\partial r^2} + \frac{a_{11}}{r} \frac{\partial u_1}{\partial r} - \frac{a_{11}}{r^2} u_1 - b_{11} u_1 + \frac{h_{11}}{r^2} \frac{\partial^2 u_1}{\partial \beta^2}$$

$$- \frac{f_{11}}{r^2} \frac{\partial v_1}{\partial \beta} + \frac{t_{11}}{r} \frac{\partial^2 v_1}{\partial r \partial \beta} + d_{11} \frac{\partial w_1}{\partial r} - c_{11} \frac{\partial w_1}{\partial r} = 0$$

(4.14a)

$$m_{11} \frac{\partial^2 v_1}{\partial r^2} + \frac{m_{11}}{r} \frac{\partial v_1}{\partial r} - \frac{m_{11}}{r^2} v_1 - n_{11} v_1 + \frac{g_{11}}{r^2} \frac{\partial^2 v_1}{\partial \beta^2}$$

$$+ \frac{t_{11}}{r} \frac{\partial^2 u_1}{\partial r \partial \beta} + \frac{f_{11}}{r^2} \frac{\partial u_1}{\partial \beta} + \frac{l_{11}}{r} \frac{\partial w_1}{\partial \beta} - \frac{k_{11}}{r} \frac{\partial w_1}{\partial \beta} = 0$$

(4.14b)

$$\frac{\partial^2 u_s}{\partial r^2} + \frac{1}{r} \frac{\partial u_s}{\partial r} - \frac{1}{r^2} u_s + \frac{1}{r} \frac{\partial^2 v_s}{\partial r \partial \beta} - \frac{1}{r^2} \frac{\partial v_s}{\partial \beta} + \frac{2}{R} \frac{\partial w_1}{\partial r}$$

$$- (1 - \mu_s) \left[\frac{1}{2r^2} \frac{\partial v_s}{\partial \beta} + \frac{1}{2r} \frac{\partial^2 v_s}{\partial r \partial \beta} - \frac{1}{2r^2} \frac{\partial^2 u_s}{\partial \beta^2} \right]$$

$$+ (1 - \mu_s) \left[\frac{u_s}{R^2} - \frac{1}{R} \frac{\partial w_1}{\partial r} \right] = - \frac{(1-\mu_s^2)}{E_s h} X \qquad (4.14c)$$

$$\frac{1}{r^2} \frac{\partial u_s}{\partial \beta} + \frac{1}{r} \frac{\partial^2 u_s}{\partial r \partial \beta} + \frac{1}{r^2} \frac{\partial^2 v_s}{\partial \beta^2} + \frac{2}{rR} \frac{\partial w_1}{\partial \beta} + (1-\mu_s) \left[\frac{1}{2r} \frac{\partial v_s}{\partial r} \right.$$

$$\left. - \frac{1}{2r^2} v_s + \frac{1}{2} \frac{\partial^2 v_s}{\partial r^2} - \frac{1}{2r} \frac{\partial^2 u_s}{\partial r \partial \beta} + \frac{1}{2r^2} \frac{\partial u_s}{\partial \beta} \right]$$

$$+ (1 - \mu_s) \left[\frac{v_s}{R^2} - \frac{1}{rR} \frac{\partial w_1}{\partial \beta} \right] = - \frac{(1 - \mu_s^2)}{E_s h} Y$$

(4.14d)

$$
r_{11} \frac{\partial^2 v_1}{\partial r^2} + \frac{r_{11}}{r^2} \frac{\partial^2 w_1}{\partial \beta^2} + \frac{r_{11}}{r} \frac{\partial w_1}{\partial r} - c_{11} v_1 - f_{11} \frac{\partial u_1}{\partial r}
$$

$$
+ o_{11} \frac{\partial u_1}{\partial r} + c_{11} \frac{u_1}{r} - f_{11} \frac{u_1}{r} - \frac{l_{11}}{r} \frac{\partial v_1}{\partial \beta}
$$

$$
+ \frac{k_{11}}{r} \frac{\partial v_1}{\partial \beta} + \frac{E_s h}{(1-\mu_s^2)} \frac{2(1+\mu_o)}{E_o} \Big[- \Big\{ \frac{2}{R} \frac{u_s}{r}
$$

$$
+ \frac{2}{R} \frac{\partial u_s}{\partial r} + \frac{2}{rR} \frac{\partial v_s}{\partial \beta} + \frac{4 v_1}{R^2} \Big\} + \frac{(1-\mu_s)}{[illegible]} \Big\{ \frac{2 w_1}{R}
$$

$$
+ \frac{u_s}{r} + \frac{\partial u_s}{\partial r} + \frac{1}{r} \frac{\partial v_s}{\partial \beta} \Big\} - \frac{h^2}{12} \Big\{ \frac{2}{rR^2} \frac{\partial w_1}{\partial r}
$$

$$
+ \frac{2}{R^2} \frac{\partial^2 w_1}{\partial r^2} + \frac{2}{R^2 r^2} \frac{\partial^2 w_1}{\partial \beta^2} + \frac{\partial^4 w_1}{\partial r^4} + \frac{2}{r} \frac{\partial^3 w_1}{\partial r^3}
$$

$$
- \frac{1}{r^2} \frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r^3} \frac{\partial w_1}{\partial r} + \frac{4}{r^4} \frac{\partial^2 w_1}{\partial \beta^2} - \frac{2}{r^3} \frac{\partial^3 w_1}{\partial r \partial \beta^2}
$$

$$
+ \frac{2}{r^2} \frac{\partial^4 w_1}{\partial r^2 \partial \beta^2} + \frac{1}{r^4} \frac{\partial^4 w_1}{\partial \beta^4} \Big\} \Big] = - \frac{2(1+\mu_o)}{E_o} Z
\tag{4.14e}
$$

where a_{11}, b_{11} etc. are given by Eqns. (4.9).

Considering loading to be anti-symmetric, the displacement functions can be written as,

$$
u_1 = U_n(r) \operatorname{Cos} (2n-1) \beta \tag{4.15a}
$$

$$
v_1 = V_n(r) \operatorname{Sin} (2n-1) \beta \tag{4.15b}
$$

$$
w_1 = W_n(r) \operatorname{Cos} (2n-1) \beta \tag{4.15c}
$$

$$u_s = U_{sn}(r) \cos (2n-1) \beta \qquad (4.15d)$$

$$v_s = V_{sn}(r) \sin (2n-1) \beta \qquad (4.15e)$$

$$w_s = W_{sn}(r) \cos (2n-1) \beta \qquad (4.15f)$$

$$(n = 1, 2, \ldots)$$

Substituting Eqns. (4.15) in Eqns. (4.14) and multiplying suitably with Cos $(2n-1)\ \beta\ d\beta$ or Sin $(2n-1)\ \beta\ d\beta$ and integrating over 0 to 2π, the resulting expressions are

$$a_{11} \frac{d^2U_n}{dr^2} + \frac{a_{11}}{r} \frac{dU_n}{dr} - \frac{a_{11}}{r^2} U_n - b_{11} U_n - \frac{(2n-1)^2}{r^2} h_{11} U_n$$

$$\frac{(2n-1)}{r^2} f_{11} V_n + \frac{(2n-1)}{r} t_{11} \frac{dV_n}{dr} + d_{11} \frac{dW_n}{dr}$$

$$c_{11} \frac{dW_n}{dr} = 0 \qquad (4.16a)$$

$$m_{11} \frac{d^2V_n}{dr^2} + \frac{m_{11}}{r} \frac{dV_n}{dr} - \frac{m_{11}}{r^2} V_n - n_{11} V_n - \frac{(2n-1)^2}{r^2} g_{11} V_n$$

$$- \frac{(2n-1)}{r} t_{11} \frac{dU_n}{dr} \cdot \frac{(2n-1)}{r^2} f_{11} U_n \quad \frac{(2n-1)}{r} l_{11} W_n$$

$$+ \frac{(2n-1)}{r} k_{11} W_n = 0 \qquad (4.16b)$$

$$\frac{1}{r} \frac{dU_{sn}}{dr} - \frac{1}{r^2} U_{sn} + \frac{d^2U_{sn}}{dr^2} + \frac{(2n-1)}{r} \frac{dV_{sn}}{dr} - \frac{(2n-1)}{dr} V_{sn}$$

$$+ \frac{2}{R} \frac{dW_n}{dr} - (1-\mu_s) \left[\frac{(2n-1)}{2r^2} V_{sn} + \frac{(2n-1)}{2r} \frac{dV_{sn}}{dr} \right.$$

$$+ \frac{(2n-1)^2}{2r^2} V_{sn}] + (1-\mu_s) [\frac{U_{sn}}{R^2} - \frac{1}{R} \frac{dW_n}{dr}]$$

$$= \frac{(1-\mu_s^2)}{E_s h \pi} \int_0^{2\pi} X \cos (2n-1)\beta \, d\beta \quad (4.16c)$$

$$- \frac{(2n-1)}{r^2} U_{sn} - \frac{(2n-1)}{r} \frac{dU_{sn}}{dr} - \frac{(2n-1)^2}{r^2} V_{sn} - \frac{2(2n-1)}{rR} W_n$$

$$+ (1-\mu_s) [\frac{1}{2r} \frac{dV_{sn}}{dr} - \frac{1}{2r^2} V_{sn} + \frac{1}{2} \frac{d^2V_{sn}}{dr^2}$$

$$+ \frac{(2n-1)}{2r} \frac{dU_{sn}}{dr} - \frac{(2n-1)}{dr} \quad \frac{(2n-1)}{2r^2} U_{sn}]$$

$$+ (1-\mu_s) [\frac{V_{sn}}{R^2} + \frac{(2n-1)}{rR} W_n]$$

$$= - \frac{(1-\mu_s)^2}{E_s h \pi} \int_0^{2\pi} Y \sin (2n-1)\beta \, d\beta \quad (4.16d)$$

$$r_{11} \frac{d^2W_n}{dr^2} + \frac{r_{11}}{r} \frac{dW_n}{dr} - \frac{(2n-1)^2}{r^2} r_{11} W_n - s_{11} W_n$$

$$- d_{11} \frac{dU_n}{dR} + c_{11} \frac{dU_n}{dr} + \frac{c_{11}}{r} U_n - \frac{d_{11}}{r} U_n$$

$$- \frac{(2n-1)}{r} l_{11} V_n + \frac{(2n-1)}{r} k_{11} V_n + \frac{E_{sh}}{(1-\mu_s^2)}$$

$$x \frac{2(1+\mu_o)}{E_o} [- \{ \frac{2}{Rr} \frac{U_{sn}}{r} + \frac{2}{R} \frac{dU_{sn}}{dr} + \frac{2(2n-1)}{rR} V_{sn}$$

$$+ \frac{4W_n}{R^2} \} + (1-\mu_s) \{ \frac{2W_n}{R^2} + \frac{U_{sn}}{rR} + \frac{1}{R} \frac{dU_{sn}}{dr}$$

$$+ \frac{(2n-1)}{rR} V_{sn} - \frac{h^2}{12} \left\{ \frac{2}{rR^2} \frac{dV_n}{dr} + \frac{2}{R^2} \frac{d^2 V_n}{dr^2} - \frac{2(2n-1)^2}{R^2 r^2} V_n \right\}$$

$$- \frac{h^2}{12} \left\{ \frac{d^4 W_n}{dr^4} + \frac{2}{r} \frac{d^3 W_n}{dr^3} - \frac{1}{r^2} \frac{d^2 W_n}{dr^2} + \frac{1}{r^3} \frac{dW_n}{dr} - \frac{4(2n-1)^2}{r^4} W_n \right.$$

$$\left. + \frac{2(2n-1)^2}{r^3} \frac{dW_n}{dr} - \frac{2}{r^3} (2n-1)^2 \frac{d^2 W_n}{dr^2} + \frac{(2n-1)^4}{r^4} W_{sn} \right\}]$$

$$= - \frac{2(1+\mu_o)}{E_o \pi} \int_0^{2\pi} Z \operatorname{Cos} (2n-1) \beta \, d\beta \qquad (4.16c)$$

where a_{11}, b_{11} etc. are given by Eqns. (4.9).

In all the examples that have been considered, it is assumed that the elastic foundation consists of single elastic material and is of finite thickness H. The dimensionless functions $\phi_1(\gamma)$, $\chi_1(\gamma)$ and $\psi_1(\gamma)$ have to be chosen from the physical constraints of the problem. In the present investigation for finite elastic layer, the following function are used,

$$\phi_1(\gamma) = \chi_1(\gamma) = \psi_1(\gamma) = \frac{H-\gamma}{H} \qquad (4.17)$$

Substituting $\phi_1(\gamma)$, $\chi_1(\gamma)$ and $\psi_1(\gamma)$ from Eqn. (4.16) in Eqn. (4.9) for a_{11}, b_{11} etc. and integrating from 0 to H, the resulting expressions are.

$$a_{11} = \frac{E_o}{(1 - \mu_o^2)} \frac{H}{3} \qquad (4.18a)$$

$$b_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{1}{H} \qquad (4.18b)$$

$$c_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{1}{2} \quad (4.18c)$$

$$d_{11} = \frac{E_o \mu_o}{(1 - \mu_o^2)} \frac{1}{2} \quad (4.18d)$$

$$r_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{H}{3} \quad (4.18e)$$

$$s_{11} = \frac{E_o}{(1 - \mu_o^2)} \frac{1}{H} \quad (4.18f)$$

$$f_{11} = \frac{(3-\mu_o) E_o}{2(1-\mu_o^2)} \frac{H}{3} \quad (4.18g)$$

$$g_{11} = \frac{E_o}{(1 - \mu_o^2)} \frac{H}{3} \quad (4.18h)$$

$$h_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{H}{3} \quad (4.18i)$$

$$k_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{1}{2} \quad (4.18j)$$

$$l_{11} = \frac{\mu_o E_o}{(1 - \mu_o^2)} \frac{1}{2} \quad (4.18k)$$

$$m_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{H}{3} \quad (4.18l)$$

$$n_{11} = \frac{E_o}{2(1 + \mu_o)} \frac{1}{H} \quad (4.18m)$$

$$t_{11} = \frac{E_o}{2(1 - \mu_o)} \frac{H}{3} \quad (4.18n)$$

To non-dimensionalise Eqns. (4.16) following dimensionless functions are assumed.

$$\bar{u}_n = U_n/h ; \quad \bar{v}_n = V_n/h ; \quad \bar{w}_n = W_n/h ; \quad \bar{u}_{sn} = U_{sn}/h ;$$

$$v_{sn} = V_{sn}/h , \quad \bar{r} = r/a \tag{4.19}$$

where a is the outer radius of the shell and h is the thickness of the shell.

If the shell is subjected to load in radial direction and moments in radial direction transferred by the column, then the components of the force X, Y and Z are equal to zero, i.e.

$$X = Y = Z = 0 \tag{4.20}$$

Keeping in view the Eqns. (4.9), (4.19) and (4.20), the differential Eqns. (4.16) can be written as given below.

Differential equations for flank, which is used in case of free shell only, are

$$\frac{d^2\bar{u}_{fn}}{d\bar{r}^2} = -\frac{1}{\bar{r}} \frac{d\bar{u}_{fn}}{d\bar{r}} + [\frac{1}{\bar{r}^2} + \frac{3(1-\mu_o)}{2} (\frac{a}{H})^2$$
$$+ (2n-1)^2 \frac{(1-\mu_o)}{2\bar{r}^2}] \bar{u}_{fn} - (2n-1) \frac{(1+\mu_o)}{2\bar{r}}$$
$$x \frac{d\bar{v}_{fn}}{d\bar{r}} + (2n-1) \frac{(3-\mu_o)}{2\bar{r}^2} \bar{v}_{fn} - \frac{3}{4} (3\mu_o-1)$$
$$x (\frac{a}{H}) \frac{d\bar{w}_{fn}}{d\bar{r}} \tag{4.21a}$$

$$\frac{d^2\bar{v}_{fn}}{d\bar{r}^2} = \frac{(2n-1)}{\bar{r}}\frac{(1+\mu_o)}{(1-\mu_o)}\frac{d\bar{u}_{fn}}{d\bar{r}} + \frac{(2n-1)}{\bar{r}^2}\frac{(3-\mu_o)}{(1-\mu_o)}\bar{u}_{fn}$$

$$- \frac{1}{\bar{r}}\frac{d\bar{v}_{fn}}{d\bar{r}} + \left[\frac{1}{\bar{r}^2} + 3\left(\frac{a}{H}\right)^2 + \frac{2(2n-1)^2}{(1-\mu_o)\bar{r}^2}\right]\bar{v}_{fn}$$

$$+ \frac{3}{2}\frac{(3\mu_o-1)}{(1-\mu_o)}(2n-1)\left(\frac{a}{H}\right)\bar{w}_{fn} \qquad (4.21b)$$

$$\frac{d^2\bar{w}_{fn}}{d\bar{r}^2} = \frac{3}{2}\frac{(3\mu_o-1)}{(1-\mu_o)}\left(\frac{a}{H}\right)\frac{d\bar{u}_{fn}}{d\bar{r}} + \frac{3}{2}\frac{(3\mu_o-1)}{(1-\mu_o)}\left(\frac{a}{H}\right)\frac{\bar{u}_{fn}}{\bar{r}}$$

$$- \frac{3}{2}\frac{(2n-1)}{\bar{r}}\frac{(3\mu_o-1)}{(1-\mu_o)}\left(\frac{a}{H}\right)\bar{v}_{fn} - \frac{1}{\bar{r}}\frac{d\bar{w}_{fn}}{d\bar{r}}$$

$$+ \left[\frac{(2n-1)^2}{\bar{r}^2} + \frac{6}{(1-\mu_o)}\left(\frac{a}{H}\right)^2\right]\bar{w}_{fn} \qquad (4.21c)$$

Differential equations for shell-foundation system are,

$$\frac{d^2\bar{u}_n}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_n}{d\bar{r}} + \left[\frac{1}{\bar{r}^2} + \frac{3(1-\mu_o)}{2}\left(\frac{a}{H}\right)^2 + [illegible]^2\right.$$

$$\left. + (2n-1)^2\frac{(1-\mu_o)}{2\bar{r}^2}\right]\bar{u}_n - (2n-1)\frac{(1+\mu_o)}{2\bar{r}}\frac{d\bar{v}_n}{d\bar{r}}$$

$$+ (2n-1)\frac{(3-\mu_o)}{2\bar{r}^2}\bar{v}_n - \frac{3}{4}(3\mu_o - 1)\left(\frac{a}{H}\right)\frac{d\bar{w}_{fn}}{d\bar{r}} \qquad (4.22a)$$

$$\frac{d^2\bar{v}_n}{d\bar{r}^2} = \frac{(2n-1)}{\bar{r}}\frac{(1+\mu_o)}{(1-\mu_o)}\frac{d\bar{u}_n}{d\bar{r}} + \frac{(2n-1)}{\bar{r}^2}\frac{(3-\mu_o)}{(1-\mu_o)}\bar{u}_n$$

$$- \frac{1}{\bar{r}}\frac{d\bar{v}_n}{d\bar{r}} + \left[\frac{1}{\bar{r}^2} + 3\left(\frac{a}{H}\right)^2 + \frac{2(2n-1)^2}{(1-\mu_o)\bar{r}^2}\right]\bar{v}_n$$

$$+ \quad \frac{3}{2}\frac{(3\mu_o-1)}{(1-\mu_o)} (2n-1) \left(\frac{a}{H}\right) \bar{v}_{fn} \qquad (4.22b)$$

$$\frac{d^2\bar{u}_{sn}}{d\bar{r}^2} = -\frac{1}{\bar{r}}\cdot\frac{d\bar{u}_{sn}}{d\bar{r}} + \left[\frac{1}{\bar{r}^2} + \frac{(2n-1)^2}{\bar{r}^2}\cdot\frac{(1-\mu_s)}{2} - (1-\mu_s)\left(\frac{a}{R}\right)^2\right]\bar{u}_{sn}$$

$$- \frac{(2n-1)}{\bar{r}}\frac{(1+\mu_s)}{2}\frac{d\bar{v}_{sn}}{d\bar{r}} + \frac{(2n-1)}{\bar{r}^2}\frac{(3-\mu_s)}{2}\bar{v}_{sn}$$

$$- (1+\mu_s)\left(\frac{a}{R}\right)\frac{d\bar{w}_n}{d\bar{r}} \qquad (4.22c)$$

$$\frac{d^2\bar{v}_{sn}}{d\bar{r}^2} = \frac{(2n-1)}{\bar{r}}\frac{(1+\mu_s)}{(1-\mu_s)}\frac{d\bar{u}_{sn}}{d\bar{r}} + \frac{(2n-1)}{\bar{r}^2}\frac{(3-\mu_s)}{(1-\mu_s)}\bar{u}_{sn}$$

$$- \frac{1}{\bar{r}}\cdot\frac{d\bar{v}_{sn}}{d\bar{r}} + \left[\frac{(2n-1)^2}{\bar{r}^2}\frac{2}{(1-\mu_s)} + \frac{1}{\bar{r}^2} - 2\left(\frac{a}{R}\right)^2\right]\bar{v}_{sn}$$

$$+ \frac{2(2n-1)}{\bar{r}}\frac{(1+\mu_s)}{(1-\mu_s)}\left(\frac{a}{R}\right)\bar{w}_{sn} \qquad (4.22d)$$

$$\frac{d^4\bar{w}_n}{d\bar{r}^4} = C\left[-\frac{(3\mu_o-1)}{2}\left(\frac{a}{R}\right)^3\left(\frac{R}{h}\right)^3\frac{d\bar{u}_n}{d\bar{r}} - \frac{(3\mu_o-1)}{2\bar{r}}\left(\frac{a}{R}\right)^3\left(\frac{R}{h}\right)^3\right.$$

$$\left. \times\ \bar{u}_n - (2n-1)\frac{(3\mu_o-1)}{2\bar{r}}\left(\frac{a}{R}\right)^3\left(\frac{R}{h}\right)^3\bar{v}_n\right]$$

$$- 12(1+\mu_s)\left(\frac{a}{R}\right)^3\left(\frac{R}{h}\right)^2\frac{d\bar{u}_{sn}}{d\bar{r}} - \frac{12(1+\mu_s)}{\bar{r}}\left(\frac{a}{R}\right)^3$$

$$\times \left(\frac{R}{h}\right)^3\bar{u}_{sn} - 12(2n-1)\frac{(1+\mu_s)}{\bar{r}}\left(\frac{a}{R}\right)^3\left(\frac{R}{h}\right)^3\bar{v}_{sn}$$

$$+ \frac{2}{\bar{r}} \frac{d^3\bar{w}_n}{d\bar{r}^3} + [2(\frac{a}{R})^2 - \frac{1}{\bar{r}^2} - \frac{2(2n-1)^2}{\bar{r}^2} - C \frac{(1-\mu_o)}{3}$$

$$- (\frac{H}{a})(\frac{R}{h})^3(\frac{a}{R})^3] \frac{d^2\bar{w}_n}{d\bar{r}^2} + [\frac{2}{\bar{r}} (\frac{a}{R})^2 + \frac{1}{\bar{r}^3}$$

$$+ \frac{2(2n-1)^2}{\bar{r}^3} - C \frac{(1-\mu_o)}{3} (\frac{H}{a})(\frac{a}{R})^3(\frac{R}{h})^3] \frac{d\bar{w}_n}{d\bar{r}}$$

$$+ [24 (1 + \mu_s) (\frac{R}{h})^2(\frac{a}{R})^4 - \frac{2(2n-1)^2}{\bar{r}^2} (\frac{a}{R})^2$$

$$- \frac{4(2n-1)^2}{\bar{r}^4} + \frac{(2n-1)^4}{\bar{r}^4} + C (1-\mu_o) \frac{(2n-1)^2}{3\bar{r}^2}$$

$$\times (\frac{H}{a})(\frac{a}{R})^3(\frac{R}{h})^3 + 2C (\frac{E}{H})(\frac{a}{R})^2 (\frac{R}{h})^3] \bar{w}_n \quad (4.22e)$$

where,

$$C = \frac{6(1-\mu_s^2)}{(1-\mu_o^2)} \frac{E_o}{E_s} \quad (4.22f)$$

Forces and moments in dimensionless form for spherical shell can be written as follows:

$$\bar{N}_1(r) = \frac{N_1(r)}{Eh} = \frac{1}{(1-\mu_s^2)}(\frac{h}{R}) [(\frac{R}{a}) \frac{d\bar{u}_{sn}}{d\bar{r}} + \bar{w}_n$$

$$+ \mu_s \left\{ (\frac{R}{a}) (2n-1) \frac{\bar{v}_{sn}}{\bar{r}} + (\frac{R}{a}) \frac{\bar{u}_{sn}}{\bar{r}} + \bar{w}_n \right\}] \quad (4.23a)$$

$$\bar{N}_2(r) = \frac{N_2(r)}{Eh} = \frac{1}{(1-\mu_s^2)} (\frac{h}{R}) [(\frac{R}{a}) \frac{\bar{u}_{sn}}{\bar{r}} + \mu_s (\frac{R}{a}) \frac{d\bar{u}_{sn}}{d\bar{r}}$$

$$+ (\frac{R}{a}) (2n-1) \frac{\bar{v}_{sn}}{\bar{r}} + (1+\mu_s) \bar{w}_n] \quad (4.23b)$$

$$\bar{M}_1(r) = \frac{M_1(r)\,a}{D} = \left(\frac{h}{R}\right)\left(\frac{R}{a}\right)\left[-\frac{\mu_s}{\bar{r}^2}(2n-1)^2\,\bar{w}_n + \frac{\mu_s}{\bar{r}}\,\frac{d\bar{w}_n}{d\bar{r}} + \frac{d^2\bar{w}_n}{d\bar{r}^2}\right] \tag{4.23c}$$

$$\bar{M}_2(r) = \frac{M_2(r)\,a}{D} = \left(\frac{h}{R}\right)\left(\frac{R}{a}\right)\left[-\frac{1}{\bar{r}^2}(2n-1)^2\,\bar{w}_n + \frac{1}{\bar{r}} \times \frac{d\bar{w}_n}{d\bar{r}} + \mu_s\,\frac{d^2\bar{w}_n}{d\bar{r}^2}\right] \tag{4.23d}$$

$$\bar{S}(r) = \frac{S(r)}{Eh} = \frac{1}{2(1+\mu_s)}\left(\frac{h}{R}\right)\left(\frac{R}{a}\right)\left[-\frac{(2n-1)}{\bar{r}}\,\bar{u}_{sn} + \frac{d\bar{v}_{sn}}{d\bar{r}} - \frac{\bar{v}_{sn}}{\bar{r}}\right] \tag{4.23e}$$

$$\bar{M}_{12}(r) = \frac{M_{12}(r)a}{D} = (1-\mu_s)\left(\frac{h}{R}\right)\left(\frac{R}{a}\right)\left[-\frac{(2n-1)}{\bar{r}^2}\,\bar{w}_n + \frac{(2n-1)}{\bar{r}}\,\frac{d\bar{w}_n}{d\bar{r}}\right] \tag{4.23f}$$

$$\bar{q}(r) = \frac{2(1+\mu_o)\;P_r(r)}{E_o} = -\frac{1}{(1-\mu_o)}\left(\frac{h}{R}\right)\left(\frac{R}{a}\right)\Bigg[-\frac{3\mu_o}{2}\,\frac{d\bar{u}_n}{d\bar{r}} - \frac{3\mu_o}{2}\,\frac{\bar{u}_n}{\bar{r}} - \frac{3\mu_o}{2}(2n-1)\,\frac{\bar{v}_n}{\bar{r}} + \frac{(1-\mu_o)}{3}\left(\frac{H}{a}\right)\frac{d^2\bar{w}_n}{d\bar{r}^2} + \frac{(1-\mu_o)}{3\bar{r}}\left(\frac{H}{a}\right)\frac{d\bar{w}_n}{d\bar{r}} - \left\{\frac{(1-\mu_o)}{3\bar{r}^2}\left(\frac{H}{a}\right)(2n-1)^2 + 2\left(\frac{a}{H}\right)\right\}\bar{w}_n\Bigg] \tag{4.23g}$$

where $M_1(r)$, $M_2(r)$ etc. are given as follows:

$$M_1(r,\beta) = M_1(r) \cos (2n-1)\beta \tag{4.24a}$$

$$M_2(r,\beta) = M_2(r) \cos (2n-1)\beta \tag{4.24b}$$

$$N_1(r,\beta) = N_1(r) \cos (2n-1)\beta \tag{4.24c}$$

$$N_2(r,\beta) = N_2(r) \cos (2n-1)\beta \tag{4.24d}$$

$$S(r,\beta) = S(r) \sin (2n-1)\beta \tag{4.24e}$$

$$M_{12}(r,\beta) = M_{12}(r) \sin (2n-1)\beta \tag{4.24f}$$

and

$$F_\gamma(r,\beta) = F_\gamma(r) \cos (2n-1)\beta \tag{4.24g}$$

$$(n = 1,2,\ldots\infty)$$

4.4.2 Thin Shallow Conical Shells on Elastic Foundations Subjected to Antisymmetric Loads:

Proceeding on similar lines, as in case of spherical shell, the governing differential equations in dimensionless form for thin shallow conical shells on elastic foundations subjected to antisymmetric loads in radial direction and radial moments transferred by the column for the foundation of single elastic material of finite thickness, can be written as given below.

Differential equations for flank which is used in case of free shell only, are

$$\frac{d^2\bar{u}_{fn}}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_{fn}}{d\bar{r}} + [\frac{1}{\bar{r}^2} + \frac{3(1-\mu_o)}{2}(\frac{a}{H})^2 +$$

$$+ \frac{(1-\mu_o)}{2\bar{r}^2}(2n-1)^2]\,\bar{u}_{fn} - (2n-1)\frac{(1+\mu_o)}{2\bar{r}}\frac{d\bar{v}_{fn}}{d\bar{r}}$$

$$+ \frac{(3-\mu_o)}{2\bar{r}^2}(2n-1)\,\bar{v}_{fn} - \frac{3(3\mu_o-1)}{1}(\frac{a}{H})\frac{d\bar{w}_{fn}}{d\bar{r}} \qquad (4.25a)$$

$$\frac{d^2\bar{v}_{fn}}{d\bar{r}^2} = \frac{(2n-1)}{\bar{r}}\frac{(1+\mu_o)}{(1-\mu_o)}\frac{d\bar{u}_{fn}}{d\bar{r}} + \frac{(2n-1)}{\bar{r}^2}\frac{(3-\mu_o)}{(1-\mu_o)}\bar{u}_{fn}$$

$$-\frac{1}{\bar{r}}\frac{d\bar{v}_{fn}}{d\bar{r}} + [\frac{1}{\bar{r}^2} + 3(\frac{a}{H})^2 + \frac{2(2n-1)^2}{(1-\mu_o)}\frac{1}{\bar{r}^2}]\,\bar{v}_{fn}$$

$$+ \frac{3(3\mu_o-1)}{2(1-\mu_o)}(2n-1)(\frac{a}{H})\,\bar{w}_{fn} \qquad (4.25b)$$

$$\frac{d^2\bar{w}_{fn}}{d\bar{r}^2} = \frac{3}{2}\frac{(3\mu_o-1)}{(1-\mu_o)}(\frac{a}{H})\frac{d\bar{u}_{fn}}{d\bar{r}} + \frac{3}{2}\frac{3(\mu_o-1)}{(1-\mu_o)}(\frac{a}{H})\frac{\bar{u}_{fn}}{\bar{r}}$$

$$+ \frac{3}{2}\frac{(2n-1)}{\bar{r}}\frac{(3\mu_o-1)}{(1-\mu_o)}(\frac{a}{H})\,\bar{v}_{fn} - \frac{1}{\bar{r}}\frac{d\bar{w}_{fn}}{d\bar{r}}$$

$$+ [\frac{(2n-1)^2}{\bar{r}^2} + \frac{6}{(1-\mu_o)}(\frac{a}{H})^2]\,\bar{w}_{fn} \qquad (4.25c)$$

Differential equations for the shell-foundation system are

$$\frac{d^2\bar{u}_n}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_n}{d\bar{r}} + [\frac{1}{\bar{r}^2} + \frac{3(1-\mu_o)}{2}(\frac{a}{H})^2 + (2n-1)^2$$

$$
\left(\frac{(1-\mu_o)}{2\bar{r}^2}\right] \bar{u}_n - (2n-1)\frac{(1+\mu_o)}{2\bar{r}}\frac{d\bar{v}_n}{d\bar{r}} + (2n-1)\frac{(3-\mu_o)}{2\bar{r}^2}\bar{v}_n
$$

$$
-\frac{3}{4}(3\mu_o\ 1)\left(\frac{a}{H}\right)\frac{d\bar{w}_{fn}}{d\bar{r}} \qquad (\text{. .26a})
$$

$$
\frac{d^2\bar{v}_n}{d\bar{r}^2} = \frac{(2n-1)}{\bar{r}}\frac{(1+\mu_o)}{(1-\mu_o)}\frac{d\bar{u}_n}{d\bar{r}} + \frac{(2n-1)}{\bar{r}^2}\frac{(3-\mu_o)}{(1-\mu_o)}\bar{u}_n
$$

$$
-\frac{1}{\bar{r}}\frac{d\bar{v}_n}{d\bar{r}} + \left[\frac{1}{\bar{r}^2} + 3\left(\frac{a}{H}\right)^2 + \frac{2(2n-1)^2}{(1-\mu_o)\bar{r}^2}\right]\bar{v}_n
$$

$$
+\frac{3}{2}\frac{(3\mu_o-1)}{(1-\mu_o)}(2n-1)\left(\frac{a}{H}\right)w_{fn} \qquad (4.26b)
$$

$$
\frac{d^2\bar{u}_{sn}}{d\bar{r}^2} = -\frac{1}{\bar{r}}\frac{d\bar{u}_{sn}}{d\bar{r}} + \left[\frac{1}{\bar{r}^2} + \frac{(1-\mu_s)}{2\bar{r}^2}(2n-1)^2\right]\bar{u}_{sn}
$$

$$
\frac{(1+\mu_s)}{2\bar{r}}(2n-1)\frac{d\bar{v}_{sn}}{d\bar{r}} + \frac{(3-\mu_s)}{2\bar{r}^2}(2n-1)\bar{v}_{sn}
$$

$$
-\frac{\cos\theta}{\bar{r}}\left[\mu_s - \frac{1}{\bar{r}}\right]\bar{v}_{sn} \qquad (4.26c)
$$

$$
\frac{d^2\bar{v}_{sn}}{d\bar{r}^2} = \frac{(1+\mu_s)}{(1-\mu_s)}\frac{(2n-1)}{\bar{r}}\frac{d\bar{u}_{sn}}{d\bar{r}} + \frac{(3-\mu_s)}{(1-\mu_s)}\frac{(2n-1)}{\bar{r}^2}\bar{u}_{sn}
$$

$$
-\frac{1}{\bar{r}}\frac{d\bar{v}_{sn}}{d\bar{r}} + \left[\frac{1}{\bar{r}^2} + \frac{2(2n-1)^2}{\bar{r}^2(1-\mu_s)}\right]\bar{v}_{sn}
$$

$$
+\frac{2\cos\theta}{(1-\mu_s)}\frac{(2n-1)}{\bar{r}^2}\bar{w}_{sn} \qquad (4.26d)
$$

Forces and moments in dimensionless form for conical shell can be written as follows:

$$\bar{N}_1(r) = \frac{N_1(r)}{Eh} = \frac{1}{(1-\mu_s^2)}\left(\frac{h}{a}\right)\left[\frac{d\bar{u}_{sn}}{d\bar{r}} + \mu_s\left\{\frac{(2n-1)}{\bar{r}}\bar{v}_{sn} + \frac{\bar{u}_{sn}}{\bar{r}} + \frac{\bar{w}_n}{\bar{r}}\cos\theta\right\}\right] \quad (4.27a)$$

$$\bar{N}_2(r) = \frac{N_2(r)}{Eh} = \frac{1}{(1-\mu_s^2)}\left(\frac{h}{a}\right)\left[\frac{(2n-1)}{\bar{r}}\bar{v}_{sn} + \frac{\bar{u}_{sn}}{\bar{r}} + \frac{\bar{w}_n}{\bar{r}}\cos\theta + \mu_s\frac{d\bar{u}_{sn}}{d\bar{r}}\right] \quad (4.27b)$$

$$\bar{M}_1(r) = \frac{M_1(r)a}{D} = \left(\frac{h}{a}\right)\left[\frac{d^2\bar{w}_n}{d\bar{r}^2} + \mu_s\left\{-\frac{(2n-1)^2}{\bar{r}^2}\bar{w}_n + \frac{1}{\bar{r}}\frac{d\bar{w}_n}{d\bar{r}}\right\}\right] \quad (4.27c)$$

$$\bar{M}_2(r) = \frac{M_2(r)a}{D} = \left(\frac{h}{a}\right)\left[-\frac{(2n-1)^2}{\bar{r}^2}\bar{w}_n + \frac{1}{\bar{r}}\frac{d\bar{w}_n}{d\bar{r}} + \mu_s\frac{d^2\bar{w}_n}{d\bar{r}^2}\right] \quad (4.27d)$$

$$\bar{N}_{12}(r) = \frac{N_{12}(r)}{Eh} = \frac{1}{2(1+\mu_s)}\left(\frac{h}{a}\right)\left[\frac{2(n-1)}{\bar{r}}\bar{u}_{sn} + \frac{d\bar{v}_{sn}}{d\bar{r}} - \frac{\bar{v}_{sn}}{\bar{r}}\right] \quad (4.27e)$$

$$\bar{M}_{12}(r) = \frac{M_{12}\,a}{D} = (1-\mu_s)\left(\frac{h}{a}\right)\left[-\frac{(2n-1)}{\bar{r}^2}\bar{w}_n + \frac{(2n-1)}{\bar{r}}\frac{d\bar{w}_n}{d\bar{r}}\right] \quad (4.27f)$$

$$\bar{q}(r) = \frac{2(1+\mu_o)\ P(r)}{\mu_o} = -\frac{1}{(1-\mu_o)}\left(\frac{h}{a}\right)\Big[-\frac{3\mu_o}{2}\frac{d\bar{u}_n}{dr}$$

$$-\frac{\mu_o}{2}\frac{\bar{u}_n}{\bar{r}} - \frac{3\mu_o}{2}(2n-1)\frac{\bar{v}_n}{\bar{r}} + \frac{(1-\mu_o)}{3}\left(\frac{H}{a}\right)$$

$$\frac{d^2\bar{w}_n}{d\bar{r}^2} + \frac{(1-\mu_o)}{3\bar{r}}\left(\frac{H}{a}\right)\frac{d\bar{w}_n}{d\bar{r}}$$

$$-\left\{\frac{(1-\mu_o)}{3\bar{r}^2}\left(\frac{H}{a}\right)(2n-1)^2 + 2\left(\frac{h}{H}\right)\right\}\bar{w}_n\Big] \qquad (4.27g)$$

where $N_1(r)$, $N_2(r)$ etc. are given by Eqns. (4.24)

4.5 BOUNDARY CONDITIONS

Three types of boundary conditions, namely simply supported, fixed and free types have been considered for both spherical and conical shells.

4.5.1 Boundary Conditions at the Outer Boundary of Shell:

(1) Simply Supported:

The following would be the boundary conditions at simply supported end

(a) $u_1 = 0$ (4.28a)

(b) $v_1 = 0$ (4.28b)

(c) $u_s = 0$ (4.28c)

(d) $v_s = 0$ (4.28d)

(e) $w_1 = 0$ (4.28e)

(f) $M_1 = 0$ (4.28f)

In dimensionless form, the above mentioned boundary conditions for both spherical and conical shells can be written as,

(a) $\bar{u}_n = 0$ (4.29a)

(b) $\bar{v}_n = 0$ (4.29b)

(c) $\bar{u}_{sn} = 0$ (4.29c)

(d) $\bar{v}_{sn} = 0$ (4.29d)

(e) $\bar{w}_n = 0$ (4.29e)

(f) $\dfrac{d^2\bar{w}_n}{d\bar{r}^2} - \dfrac{(2n-1)^2}{\bar{r}^2}\mu_s \bar{w}_n + \dfrac{\mu_s}{\bar{r}}\dfrac{d\bar{w}_n}{d\bar{r}} = 0$ (4.29f)

(2) Fixed or Built-in:

The boundary conditions for fixed end are:

(a) $u_1 = 0$ (4.30a)

(b) $v_1 = 0$ (4.30b)

(c) $u_s = 0$ (4.30c)

(d) $v_s = 0$ (4.30d)

(e) $w_1 = 0$ (4.30e)

(f) $\dfrac{dw_1}{dr} = 0$ (4.30f)

In dimensionless form, the above mentioned boundary conditions for both spherical and conical shell can be written as,

(a) $\bar{u}_n = 0$ (4.31a)

(b) $\bar{v}_n = 0$ (4.31b)

(c) $\bar{u}_{sn} = 0$ (4.31c)

(d) $\bar{v}_{sn} = 0$ (4.31d)

(e) $\bar{w}_{n} = 0$ (4.31e)

(f) $\dfrac{d\bar{w}_n}{d\bar{r}} = 0$ (4.31f)

(3) Free Case:

Boundary conditions at the far end of the flank can be written as

(a) $u_f = 0$ (4.32a)

(b) $v_f = 0$ (4.32b)

(c) $w_f = 0$ (4.32c)

Boundary conditions at the free end of the shell can be written as,

(a) $u_1 = u_f$ (4 33a)

(b) $v_1 = v_f$ (4.33b)

(c) $w_1 = w_f$ (4.33c)

(d) Normal force in flank

= Normal force in foundation (4.33d)

(e) Shear force inf flank

= Shear force in foundation (4.33e)

(f) Normal force in shell = 0 (4.33f)

(g) Shear force in shell = 0 (4.33g)

(h) Transverse shear in shell

+ Transverse shear in foundation

= Transverse shear in flank (4.33h)

(i) $M_1 = 0$ (4.33i)

Above mentioned boundary conditions in dimensionless form for spherical shell can be written as given below.

Boundary conditions at the far end of the flank are

(a) $\bar{u}_{fn} = 0$ (4.34a)

(b) $\bar{v}_{fn} = 0$ (4.34b)

(c) $\bar{w}_{fn} = 0$ (4.34c)

Boundary conditions at the free end are

(a) $\bar{u}_n = \bar{u}_{fn}$ (4.35a)

(b) $\bar{v}_n = \bar{v}_{fn}$ (4.35b)

(c) $\bar{w}_n = \bar{w}_{fn}$ (4.35c)

$$\text{(d)} \quad \frac{d\bar{u}_{fn}}{d\bar{r}} - \frac{d\bar{u}_n}{d\bar{r}} + \mu_o \left[\frac{\bar{u}_{fn}}{\bar{r}} - \frac{\bar{u}_n}{\bar{r}} + \frac{(2n-1)}{\bar{r}} \bar{v}_{fn} - \frac{(2n-1)}{\bar{r}} \bar{v}_n + \frac{3}{2} \left(\frac{a}{H}\right) \bar{w}_{fn} - \frac{3}{2} \left(\frac{a}{H}\right) \bar{w}_n \right] = 0 \tag{4.35d}$$

$$\text{(e)} \quad - \frac{(2n-1)}{\bar{r}} \bar{u}_{fn} + \frac{(2n-1)}{\bar{r}} \bar{u}_n + \frac{d\bar{v}_{fn}}{d\bar{r}} - \frac{d\bar{v}_n}{d\bar{r}} - \frac{\bar{v}_{fn}}{\bar{r}} + \frac{\bar{v}_n}{\bar{r}} = 0 \tag{4.35e}$$

$$\text{(f)} \quad \frac{d\bar{u}_{sn}}{d\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_n + \mu_s \left[\frac{(2n-1)}{\bar{r}} \bar{v}_{sn} + \frac{\bar{u}_{sn}}{\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_n\right] = 0 \tag{4.35f}$$

$$(g) \quad \frac{d\bar{v}_{sn}}{d\bar{r}} - \frac{\bar{v}_{sn}}{\bar{r}} - \frac{(2n-1)}{\bar{r}} \bar{u}_{sn} = 0 \qquad (4.35g)$$

$$(h) \quad \frac{6(1-\mu_s^2)}{(1+\mu_o)} \frac{E_o}{E_s} \left[\frac{1}{2} \bar{u}_{fn} - \frac{1}{2} \bar{u}_n + \frac{1}{3} \left(\frac{H}{a}\right) \frac{d\bar{w}_{fn}}{d\bar{r}} \right.$$

$$\left. - \frac{1}{3} \left(\frac{H}{a}\right) \frac{d\bar{w}_n}{d\bar{r}} \right] + \left(\frac{h}{R}\right)^3 \left(\frac{R}{a}\right)^3 \left[\frac{d^3\bar{w}_n}{d\bar{r}^3} + \frac{1}{\bar{r}} \frac{d^2\bar{w}_n}{d\bar{r}^2} \right.$$

$$\left. - \frac{1}{\bar{r}^2} \frac{d\bar{w}_n}{d\bar{r}} - \frac{(2n-1)^2}{\bar{r}^2} \frac{d\bar{w}_n}{d\bar{r}} + \frac{2}{\bar{r}^3} (2n-1)^2 \bar{w}_n \right] = 0$$

(4.35h)

$$(i) \quad \frac{d^2\bar{w}_n}{d\bar{r}^2} + \frac{\mu_s}{\bar{r}} \frac{d\bar{w}_n}{d\bar{r}} - (2n-1)^2 \frac{\mu_s}{\bar{r}^2} \bar{w}_n = 0 \qquad (4.35i)$$

Boundary conditions in free case for conical shell in dimensionless form can be written as given below.

At the far end of the flank the boundary conditions are,

$$(a) \quad \bar{u}_{fn} = 0 \qquad (4.36a)$$

$$(b) \quad \bar{v}_{fn} = 0 \qquad (4.36b)$$

$$(c) \quad \bar{w}_{fn} = 0 \qquad (4.36c)$$

Boundary conditions at the free end of the shell are,

$$(a) \quad \bar{u}_n = \bar{u}_{fn} \qquad (4.37a)$$

$$(b) \quad \bar{v}_n = \bar{v}_{fn} \qquad (4.37b)$$

$$(c) \quad \bar{w}_n = \bar{w}_{fn} \qquad (4.37c)$$

$$(d)\quad \frac{d\bar{u}_{fn}}{d\bar{r}} - \frac{d\bar{u}_n}{d\bar{r}} + \frac{\mu_o}{\bar{r}}(2n-1)\,\bar{v}_{fn} - \frac{\mu_o}{\bar{r}}(2n-1)\,\bar{v}_n$$

$$+ \frac{\mu_o}{r}\,\bar{u}_{fn} - \frac{\mu_o}{r}\,\bar{u}_n + \frac{3\mu_o}{2}\left(\frac{a}{H}\right)\bar{w}_{fn}$$

$$- \frac{3}{2}\,\mu_o\left(\frac{a}{H}\right)\bar{w}_n = 0 \qquad (4.37d)$$

$$(e)\quad -(2n-1)\,\frac{\bar{u}_{fn}}{\bar{r}} + (2n-1)\,\frac{\bar{u}_n}{\bar{r}} + \frac{d\bar{v}_{fn}}{d\bar{r}} - \frac{d\bar{v}_n}{d\bar{r}} - \frac{\bar{v}_{fn}}{\bar{r}}$$

$$+ \frac{\bar{v}_n}{\bar{r}} = 0 \qquad (4.37e)$$

$$(f)\quad \frac{d\bar{u}_{sn}}{d\bar{r}} + \mu_s\left[(2n-1)\,\frac{\bar{v}_{sn}}{\bar{r}} + \frac{\bar{u}_{sn}}{\bar{r}} + \frac{\bar{w}_n}{\bar{r}}\,\mathrm{Cos}\;\theta\right] = 0 \qquad (4.37f)$$

$$(g)\quad -(2n-1)\,\frac{\bar{u}_{sn}}{\bar{r}} + \frac{d\bar{v}_{sn}}{d\bar{r}} - \frac{\bar{v}_{sn}}{\bar{r}} = 0 \qquad (4.37g)$$

$$(h)\quad \frac{12(1-\mu_s{}^2)}{2(1+\mu_o)}\,\frac{E_o}{E_s}\left(\frac{a}{h}\right)^3\left[\frac{\bar{u}_n}{2} - \frac{\bar{u}_{fn}}{2} + \frac{1}{3}\left(\frac{H}{a}\right)\frac{d\bar{w}_n}{d\bar{r}}\right.$$

$$\left. - \frac{1}{3}\left(\frac{H}{a}\right)\frac{d\bar{w}_{fn}}{d\bar{r}}\right] - \frac{d^3\bar{w}_n}{d\bar{r}^3} - \frac{1}{\bar{r}}\,\frac{d^2\bar{w}_n}{d\bar{r}^2} + \frac{1}{\bar{r}^2}\,\frac{d\bar{w}_n}{d\bar{r}}$$

$$+ \frac{(2n-1)^2}{\bar{r}^2}\,\frac{d\bar{w}_n}{d\bar{r}} - \frac{2(2n-1)^2}{\bar{r}^3}\,\bar{w}_n = 0 \qquad (4.37h)$$

$$(i)\quad \frac{d^2\bar{w}_n}{d\bar{r}^2} + \frac{\mu_s}{\bar{r}}\,\frac{d\bar{w}_n}{d\bar{r}} - \frac{\mu_s(2n-1)^2}{\bar{r}^2}\,\bar{w}_n = 0 \qquad (4.37i)$$

4.5.2 Boundary Conditions at the Column End:

The antisymmetric load distribution along the column has been taken as,

$$q = \sum_{n=1}^{\infty} \frac{P}{\pi b} \operatorname{Cos} \, (2n-1)\beta \qquad (4.38)$$

where P is either radial load or moment in radial direction and b is the radius of the column. If the column is assumed to be very stiff or rigid in comparison to shell, the load distribution along the column can be taken as,

$$q = \frac{P}{\pi b} \operatorname{Cos} \beta \qquad (4.39)$$

since no higher order mode shapes are possible.

Boundary conditions at the column end can be written in three ways. Firstly by assuming that the foundation core (portion of foundation below the column) is rigid or very stiff. Secondly by assuming that the core of foundation is of negligible stiffness and lastly by assuming a linear combination of the above two extremes. To be on safer side second criterion has been considered in writing the boundary conditions.

4.5.2.1 Load in Radial Direction:

Boundary conditions at the column end of the shell for loads in radial direction can be written as;

(a) Normal force in shell (N_1)

= Load per unit length (4.40a)

(b) $\frac{d\bar{w}_n}{d\bar{r}} = 0$ (4.40b)

(c) Transverse shear in shell

+ Transverse shear in foundation = 0 (4.40c)

(d) Normal force in foundation = 0 (4.40d)

(e) Shear force in Shell = 0 (4.40e)

(f) Shear force in foundation = 0 (4.40f)

In dimensionless form the above mentioned boundary conditions for spherical shell can be written as,

(a)
$$\frac{d\bar{u}_{sn}}{d\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_n + \mu \left[\frac{(2n-1)}{\bar{r}} \bar{v}_{sn} + \frac{\bar{u}_{sn}}{\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_n\right]$$
$$\text{Cos } (2n-1)\ \beta = \frac{(1-\mu_s^{\ 2})}{\pi} \left(\frac{a}{b}\right) \frac{P}{E_s h^2} \text{ Cos } \beta \quad (4.41a)$$

From orthogonal properties of cosine function, it can be shown that,

$$\frac{d\bar{u}_{sn}}{d\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_n + \mu_s \left[(2n-1) \frac{\bar{v}_{sn}}{\bar{r}} + \frac{\bar{u}_{sn}}{\bar{r}} + \left(\frac{a}{R}\right) \bar{w}_n\right]$$
$$= \frac{(1-\mu_s^{\ 2})}{\pi} \left[\left(\frac{a}{b}\right)\left(\frac{P}{E_s h^2}\right)\right] ; \quad n = 1$$
$$= 0 \qquad ; \quad n \geq 2 \qquad (4.41b)$$

$$\text{(b)} \quad \frac{d\bar{w}_n}{d\bar{r}} = 0 \qquad (4.41c)$$

$$\text{(c)} \quad (\frac{h}{R})^3 \ (\frac{R}{a})^3 \ [\frac{d^3\bar{w}_n}{d\bar{r}^3} + \frac{1}{\bar{r}} \frac{d^2\bar{w}_n}{d\bar{r}^2} - \frac{1}{\bar{r}^2} \frac{d\bar{w}_n}{d\bar{r}}$$

$$- \frac{(2n-1)^2}{\bar{r}^2} \frac{d\bar{w}_n}{d\bar{r}} + \frac{3}{\bar{r}^3} (2n-1)^2 \ \bar{w}_n]$$

$$- \frac{12(1-\mu_s^{\ 2})}{2(1 + u_s)} \frac{E_o}{E_s} [\frac{\bar{u}_n}{2} + \frac{1}{3} (\frac{H}{a}) \frac{d\bar{w}_n}{d\bar{r}}] = 0 \qquad (4.41d)$$

$$\text{(d)} \quad \frac{d\bar{u}_n}{d\bar{r}} + \frac{\mu_o(2n-1)}{\bar{r}} \bar{v}_n + \mu_o \frac{\bar{u}_n}{\bar{r}} + \frac{3\mu_o}{2} (\frac{a}{H}) \bar{w}_n = 0 \qquad (4.41e)$$

$$\text{(e)} \quad - \frac{(2n-1)}{\bar{r}} \bar{u}_{sn} + \frac{d\bar{v}_{sn}}{d\bar{r}} - \frac{\bar{v}_{sn}}{\bar{r}} = 0 \qquad (4.41f)$$

$$\text{(f)} \quad - \frac{(2n-1)}{\bar{r}} \bar{u}_n + \frac{d\bar{v}_n}{d\bar{r}} - \frac{\bar{v}_n}{\bar{r}} = 0 \qquad (4.41g)$$

For conical shells, boundary conditions in dimensionless form can be written as,

$$\text{(a)} \quad \frac{d\bar{u}_{sn}}{d\bar{r}} + \mu_s \frac{\bar{u}_{sn}}{\bar{r}} + (2n-1) \ \mu_s \frac{\bar{v}_{sn}}{\bar{r}} + \mu_s \text{ Cos } \theta \ \frac{\bar{w}_n}{\bar{r}}$$

$$= \frac{(1-\mu_s^{\ 2})}{\pi} (\frac{a}{b})(\frac{P}{E_s h^2}) \qquad n = 1$$

$$= 0 \qquad n \geq 2 \qquad (4.42a)$$

$$\text{(b)} \quad \frac{dw_n}{d\bar{r}} = 0 \qquad (4.42b)$$

$$(c) \quad -\frac{d^3\bar{w}_n}{d\bar{r}^3} - \frac{1}{\bar{r}}\frac{d^2\bar{w}_n}{d\bar{r}^2} + \frac{1}{\bar{r}^2}\frac{d\bar{w}_n}{d\bar{r}} + \frac{(2n-1)^2}{\bar{r}^2}\frac{d\bar{w}_n}{d\bar{r}}$$

$$-\frac{2(2n-1)^2}{\bar{r}^3}\bar{w}_n + \frac{6(1-\mu_s^2)}{(1+\mu_o)}\frac{E_o}{E_s}\left(\frac{a}{h}\right)^3\left[\frac{\bar{u}_n}{2}\right.$$

$$\left. + \frac{1}{3}\left(\frac{H}{a}\right)\frac{d\bar{w}_n}{d\bar{r}}\right] = 0 \qquad (4.42c)$$

$$(d) \quad \frac{d\bar{u}_n}{d\bar{r}} + \mu_o(2n-1)\frac{\bar{v}_n}{\bar{r}} + \frac{\mu_o}{\bar{r}}\bar{u}_n + \frac{3}{2}\mu_o\left(\frac{a}{H}\right)\bar{w}_n = 0 \qquad (4.42d)$$

$$(e) \quad -\frac{(2n-1)}{\bar{r}}\bar{u}_{sn} + \frac{d\bar{v}_{sn}}{d\bar{r}} - \frac{\bar{v}_{sn}}{\bar{r}} = 0 \qquad (4.42e)$$

$$(f) \quad -\frac{(2n-1)}{\bar{r}}\bar{u}_n + \frac{d\bar{v}_n}{d\bar{r}} - \frac{\bar{v}_n}{\bar{r}} = 0 \qquad (4.42f)$$

4.5.2.2 Moment in Radial Direction:

Here again only first mode is effective. Except for boundary conditions (a) and (b) of Eqns. (4.40) all remaining four boundary conditions are same as in case of load in radial direction.

These two boundary conditions are,

(a) $M_1 = M_{applied}$ in per unit length (4.43a)

(b) Normal force in shell $= 0$ (4.43b)

For spherical shell, the two boundary conditions in dimensionless form can be written as

$$(a) \quad (\frac{R}{a})(\frac{h}{R}) \left[\frac{d^2\bar{w}_n}{d\bar{r}^2} + \mu_s \left\{ - \frac{(2n-1)^2}{\bar{r}^2} \bar{w}_n + \frac{1}{\bar{r}} \frac{d\bar{w}_n}{d\bar{r}} \right\} \right]$$

$$= - \frac{1}{\pi} \left[(\frac{a}{b}) \frac{M}{D} \right] \quad ; \quad n = 1$$

$$= 0 \qquad n > 0 \qquad (4.44a)$$

$$(b) \quad \frac{d\bar{u}_{sn}}{d\bar{r}} + (\frac{a}{R}) \bar{w}_n + \mu_s \left\{ \frac{(2n-1)}{\bar{r}} \bar{v}_{sn} + \frac{\bar{u}_{sn}}{\bar{r}} + (\frac{a}{R}) \bar{w}_n \right\}]$$

$$= 0 \qquad (4.44b)$$

where M is the applied moment,

D is the rigidity modulus.

$(\frac{a}{b})(\frac{M}{D})$ has been taken as the load parameters.

Similarly for conical shells these two boundary conditions are

$$(a) \quad (\frac{b}{a}) \left[\frac{d^2\bar{w}_n}{d\bar{r}^2} + \mu_s \left\{ - \frac{(2n-1)^2}{\bar{r}^2} \bar{w}_n + \frac{1}{\bar{r}} \frac{d\bar{w}_n}{d\bar{r}} \right\} \right]$$

$$= - \frac{1}{\pi} \left[(\frac{a}{b})(\frac{M}{D}) \right] \quad ; n = 1$$

$$= 0 \qquad ; n = 2 \qquad (4.45a)$$

$$(b) \quad \frac{d\bar{u}_{sn}}{d\bar{r}} + \mu_s \frac{\bar{u}_{sn}}{\bar{r}} + (2n-1) \mu_s \frac{\bar{v}_{sn}}{\bar{r}} + \mu_s \cos\theta \frac{\bar{w}_n}{\bar{r}} = 0$$

$$(4.45b)$$

4.6 NUMERICAL CALCULATIONS AND CONCLUSIONS:

In all the numerical examples that have been solved, following parameter values have been kept constant.

$$\frac{H}{2} = 2.00$$

$$\frac{b}{a} = 0.2$$

$$\mu_s = 0.25$$

$$\mu_o = 0.17$$

In case of spherical shell, $(\frac{R}{a}) = 4.00$ has been kept constant while $(\frac{h}{R})$ and $(\frac{E_s}{E_o})$ have been varied. In second set, $(\frac{h}{R}) = 0.0333$ has been kept constant while $(\frac{k}{a})$ and $(\frac{E_s}{E_o})$ have been varied. Similarly in case of conical shells $(\frac{a}{h}) = 14.0$ is a constant while θ and $(\frac{E_s}{E_o})$ take different values. Results have been presented in plots drawn between $\bar{r}$ and functional values at unit load parameters. Load parameter in case of radial load is taken as,

$$\tilde{P} = (\frac{a}{b})(\frac{P}{E_s h^2}) \qquad (4.46)$$

and in case of moment is taken as

$$\tilde{P} = (\frac{a}{b})(\frac{M}{D}) \qquad (4.47)$$

From boundary conditions at column end it can be concluded that only $n = 1$ gives rise to response in structure. For $n \geq 2$, there would not be any loading on structure, hence there would be no response of the structure.

Keeping this in mind, response functions per unit value of load parameter have been defined as [writing n = 1]

$$\tilde{u}_1 = \bar{u}_1/\bar{P} \quad (4.48a)$$

$$\tilde{u}_s = \bar{u}_{s1}/\bar{P} \quad (4.48b)$$

$$\tilde{w}_1 = \bar{w}_1/\bar{P} \quad (4.48c)$$

$$\tilde{N}_1 = \bar{N}_1/\bar{P} \quad (4.48d)$$

$$\tilde{N}_2 = \bar{N}_2/\bar{P} \quad (4.48e)$$

$$\tilde{M}_1 = \bar{M}_1/\bar{P} \quad (4.48f)$$

$$\tilde{M}_2 = \bar{M}_2/\bar{P} \quad (4.48g)$$

and

$$\tilde{q} = \bar{q}/\bar{P} \quad (4.48h)$$

4.6.1 <u>Load in Radial Direction</u>:

4.6.1.1 Spherical Shell:

(a) Simply Supported and Fixed Boundary Conditions:

As shown in Figs. (4.1a), (4.1b), (4.2a) and (4.2b) responses $\tilde{u}_s$, $\tilde{w}_1$ increase with decrease in $(\frac{h}{R})$ also maximum value of responses $\tilde{u}_s$, $\tilde{w}_1$ in fixed case are slightly lower than those in simply supported case. $\tilde{N}_1$ remains positive all along $\tilde{r}$ and increases with increase in $(\frac{h}{R})$ as shown in Figs. (4.1c) and (4.2c). $\tilde{M}_r$ for simply supported case is shown in Figs. (4.1d) whereas $\tilde{q}$ is shown in Figs. (4.1e).

As shown in Figs. (4.3a), (4.4a) values of responses $\tilde{u}_s$ increase with decrease in $(\frac{R}{a})$. $\tilde{w}_1$ decreases with increase in $(\frac{h}{R})$ and the curve flattens for higher values of $(\frac{R}{a})$ which shows the tendency of $\tilde{w}$ going to zero along $\bar{r}$ for large value of $(\frac{R}{a})$, that is, in case of a plate [Fig. (4.3b), (4.4b)]. $\tilde{N}_1$ remains positive for all values of $\tilde{r}$ and increases with increase in $(\frac{R}{a})$ and tends to become maximum for a plate [Fig. (4.3c), (4.4c)]. From Figs. (4.3d) and (4.4c), it can be seen that $\tilde{q}$ decreases with increase of $(\frac{R}{a})$ and it tends to zero along $\bar{r}$ as $(\frac{R}{a})$ takes higher values. Similar conclusions can be drawn from Fig. (4.4d) showing variation of $\bar{M}_1$ along r.

(b) Free Boundary Condition:

$\tilde{w}_1'$ is zero at the far end of the flank and it is negative near the free boundary and becomes positive away from the free boundary which would obviously be the result in case of free shells [Figs. (4.5b), (4.6a)]. $\tilde{w}$ approaches zero along $\bar{r}$ for larger values of $(\frac{R}{a})$ as can be seen from Fig. (4.6a). $\tilde{q}$ is slightly negative near the free boundary and positive in the remaining portion [Figs. (4.5a), (4.6b)]. It can also be observed that $\tilde{q}$ tends to zero along $\bar{r}$ for large value of $(\frac{R}{a})$.

4.6.1.2 Conical Shell:

Simply Supported and Fixed Boundary Conditions:

In Figs. (4.7a), (4.8a) variations of $\tilde{u}_s$ have been shown for different values of Θ for simply supported and fixed boundary conditions. Variation of $\tilde{M}_1$ has been shown in Figs. (4.7b) and (4.8b) for different values of Θ. $\tilde{M}_1$ remains positive along $\bar{r}$ and is maximum at the column end where it is equal to the prescribed $\tilde{M}_1$. $\tilde{M}_1$, as shown in Figs. (4.7c) and (4.8c), is zero along $\bar{r}$ when $\Theta = \frac{\pi}{2}$ (plate), which would obviously be the case since no moment can develope if plate is subjected to inplane forces.

4.6.2 Moment in Radial Direction:

4.6.2.1 Spherical Shell

(a) Simply Supported and Fixed Boundary Conditions:

$\tilde{w}_1$, as shown in Fig. (4.10a), remains positive all along $\bar{r}$ and the value of $\tilde{w}_1$ increases with decrease of $(\frac{h}{R})$ ratio for fixed case. Similar is the case for simply supported boundary condition (Figure is not shown). Variation of $\tilde{M}_1$ along $\bar{r}$ is almost same for both simply supported and fixed boundary conditions and for different $(\frac{h}{R})$ values, except near the boundary where $\bar{M}_r$ is zero in case of simply supported shell and is having slight positive value in fixed core. $\tilde{M}_1$ is maximum near the column end where its

value is equal to the prescribed moment [Figs. (4.9), (4.10b)].

$\tilde{w}_1$ is shown in Fig. (4.12a) for different values of $(\frac{R}{a})$. $\tilde{M}_1$ is again having almost the same variation along $\bar{r}$ for all values of $(\frac{R}{a})$ considered and for both simply supported and fixed types of boundary conditions (except near the boundary). Only $\tilde{M}_1$ for fixed boundary condition has been shown in Fig. (4.12b). $\tilde{M}_2$ for simply supported case has been shown in Fig. (4.11). It can be seen that $\tilde{M}_2$ increases with increase in $(\frac{R}{a})$ ratio.

(b) Free Boundary Condition:

Fig. (4.13a) shows the variation of $\tilde{w}_1$ along $\bar{r}$ for different values of $(\frac{h}{R})$. It can be seen that $\tilde{w}_1$ increases slowly from zero at the far end of the flank to some small positive value at free boundary of the shell and goes negative away from the boundary. Fig. (4.14a) shows the variation of $\tilde{w}_1$ along $\bar{r}$ for different values of $(\frac{R}{a})$. Similar observations, as has been made above, can also be made. Variation of $\tilde{M}_1$ along $\bar{r}$ is same (or very near that) for all $(\frac{h}{R})$ [Fig. (4.13b)] variation of $\tilde{M}_2$ along $\bar{r}$ for different $(\frac{R}{a})$ is shown in Fig. (4.14b). It can be observed that $\tilde{M}_2$ increases with increase in $(\frac{R}{a})$.

4.6.2.2 Conical Shell:

(a) Simply supported and fixed boundary conditions:

Variation of $\tilde{W}_1$ along $\bar{r}$ has been shown in Figs. (4.15a) and (4.16) for different values of θ. $\tilde{W}_1$ is maximum for $\theta = \frac{\pi}{2}$ (plate). As shown in Figs. (4.14b), $\tilde{M}_1$ is almost same for all values of θ considered. Similar is the case for fixed boundary condition (Figure is not shown). $\tilde{M}_1$ attains maximum value at the column end where it is equal to the prescribed value of $\tilde{M}_1$.

(b) Free boundary condition:

Fig. (4.17) shows the variation of $\tilde{w}_1$ along $\bar{r}$ for different values of θ. It can be observed from the plot that $\tilde{w}_1$ is zero at the far end of the flank and slowly increases to some positive value near the free boundary and remains positive over a large portion of shell and rapidly drops to a negative value near the column end. This negative value is much larger than its positive value. At $\theta = \frac{\pi}{2}$, $\tilde{w}$ has maximum negative value.

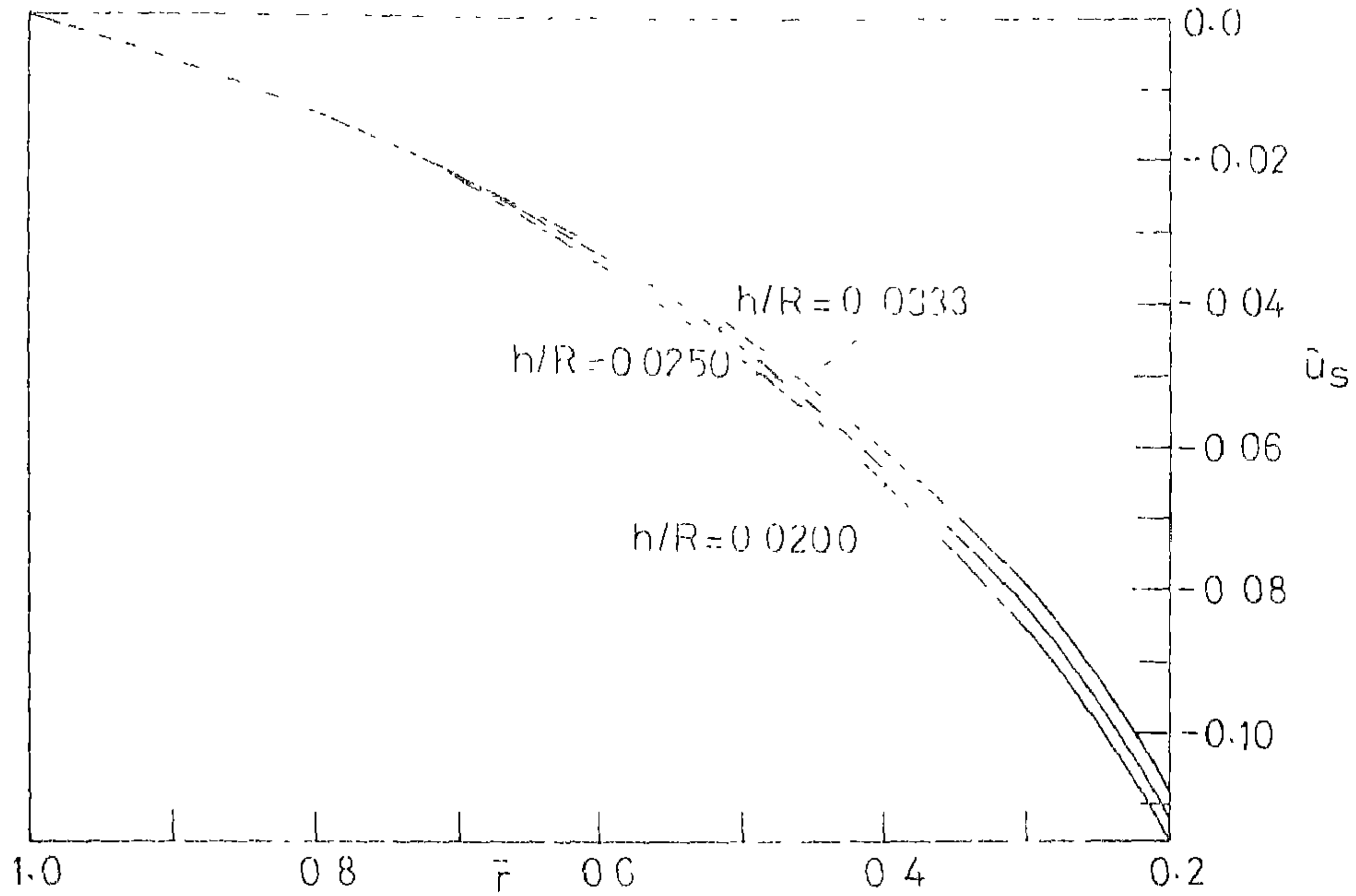


FIG. 4.1a SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\bar{u}_s$ FOR DIFFERENT (h/R) VALUES

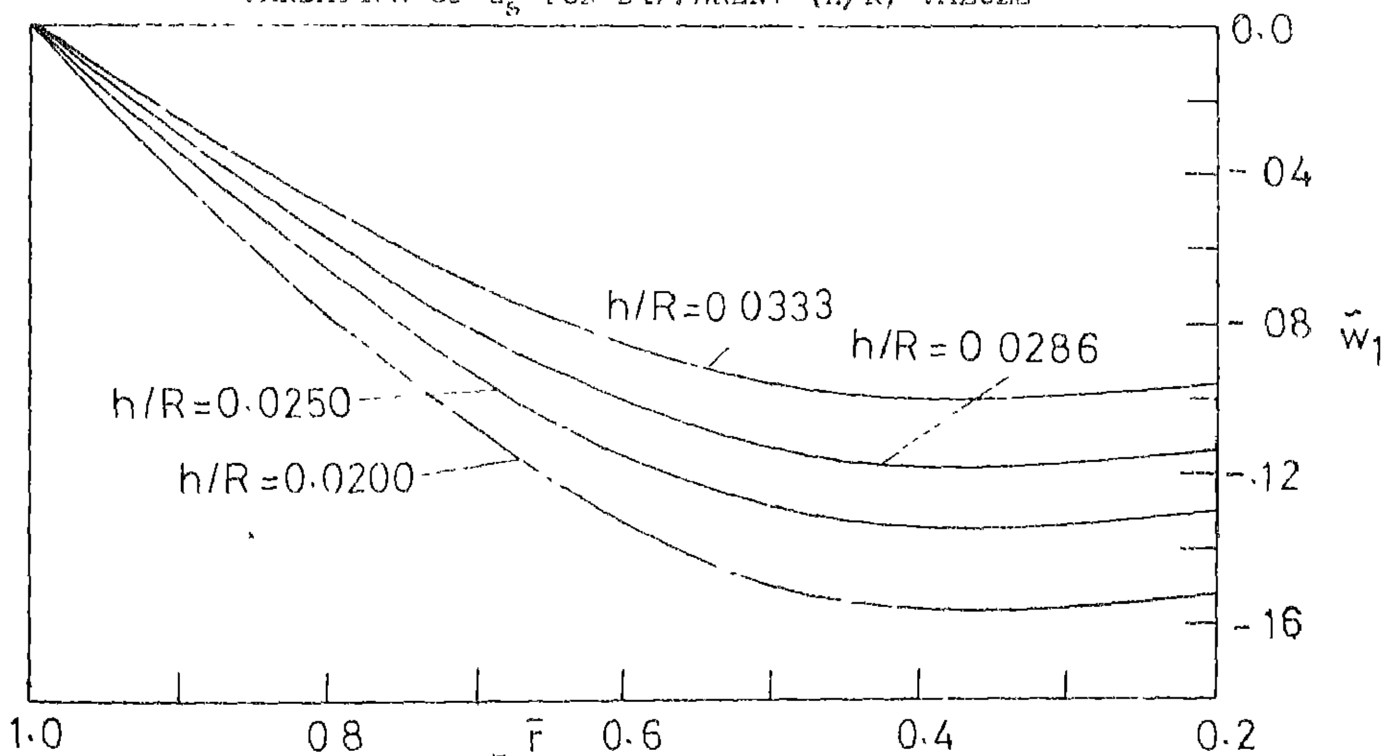


FIG. 4.1b SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD

VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (h/R) VALUES

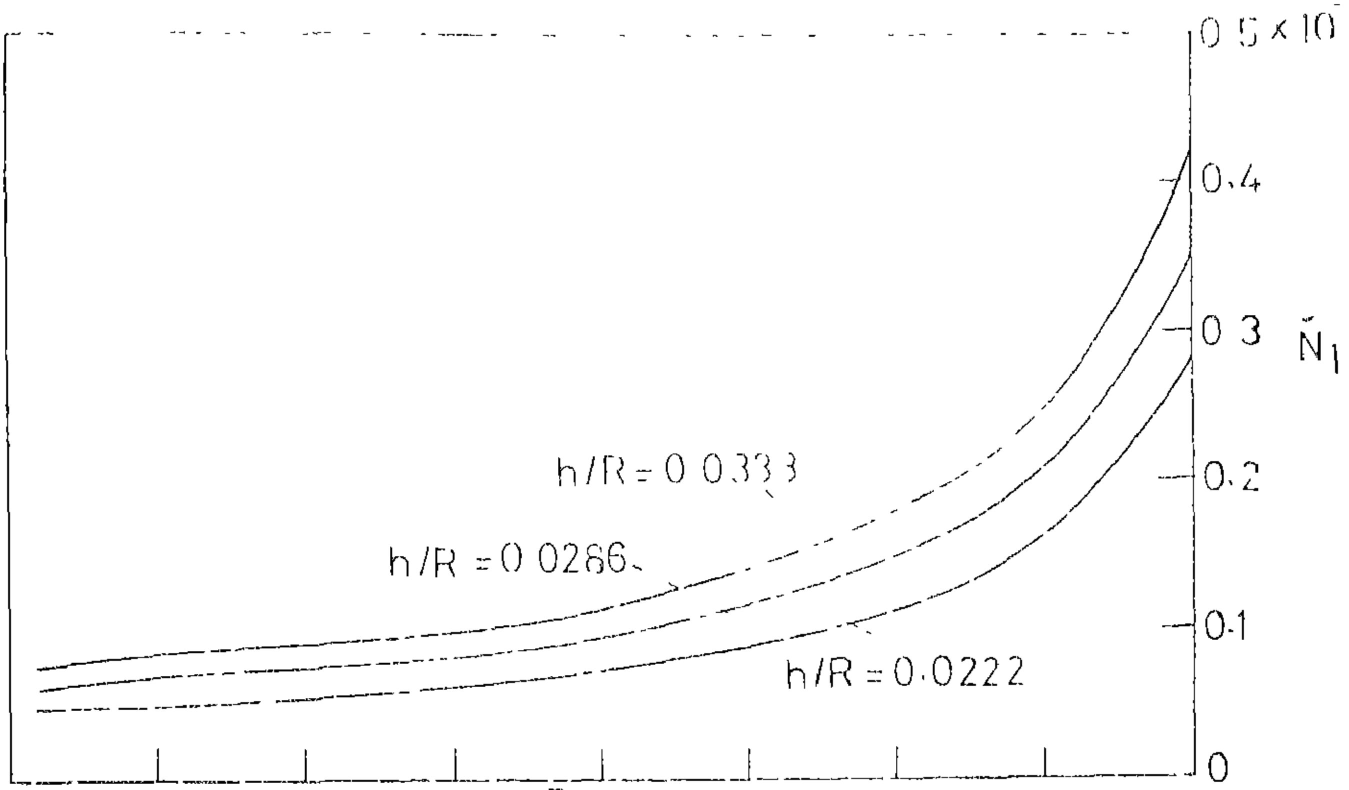


FIG. 4.1c SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\bar{N}_1$ FOR DIFFERENT (h/R) VALUES

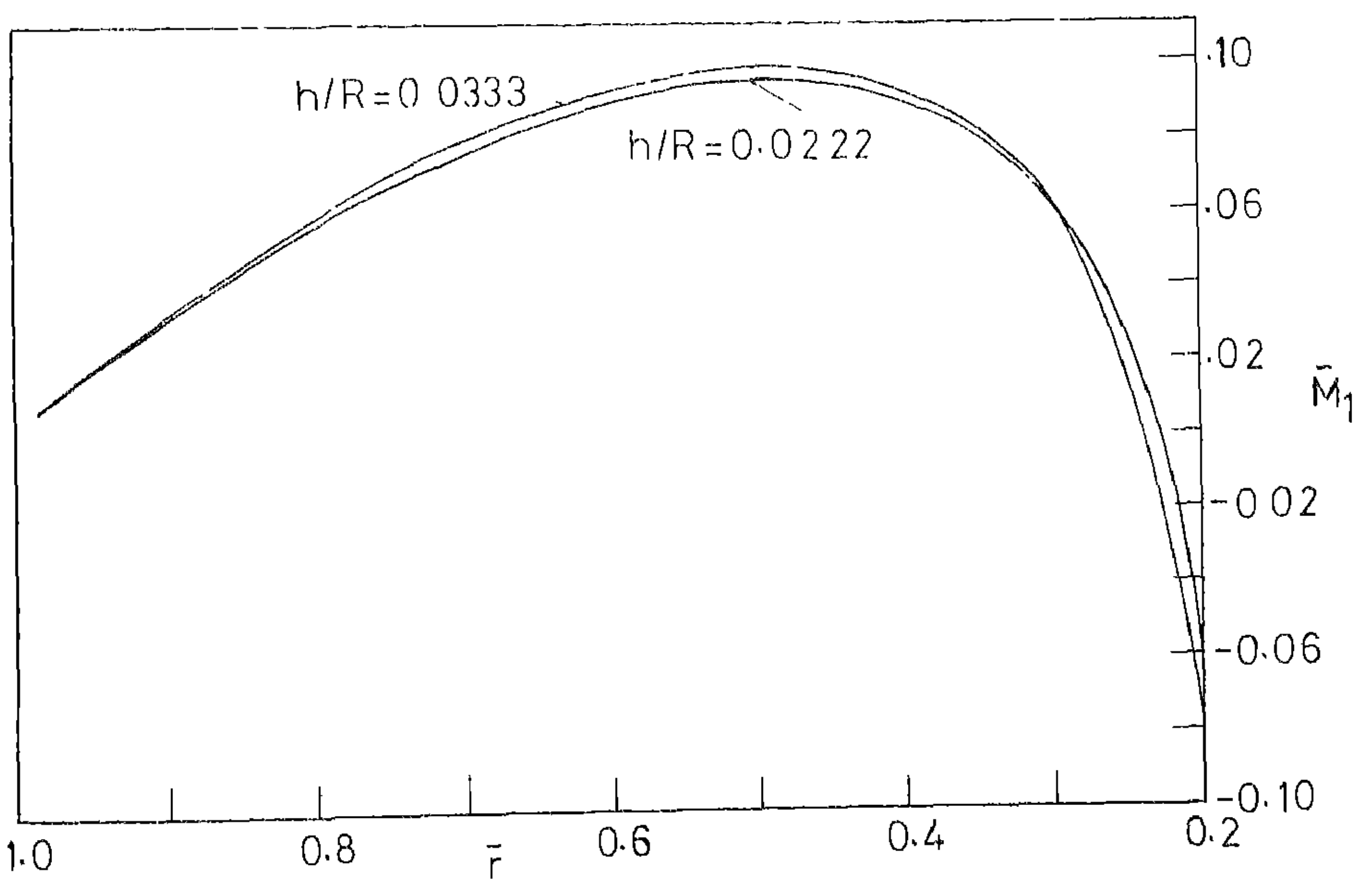


FIG. 4.1d SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\bar{M}_1$ FOR DIFFERENT (h/R) VALUES

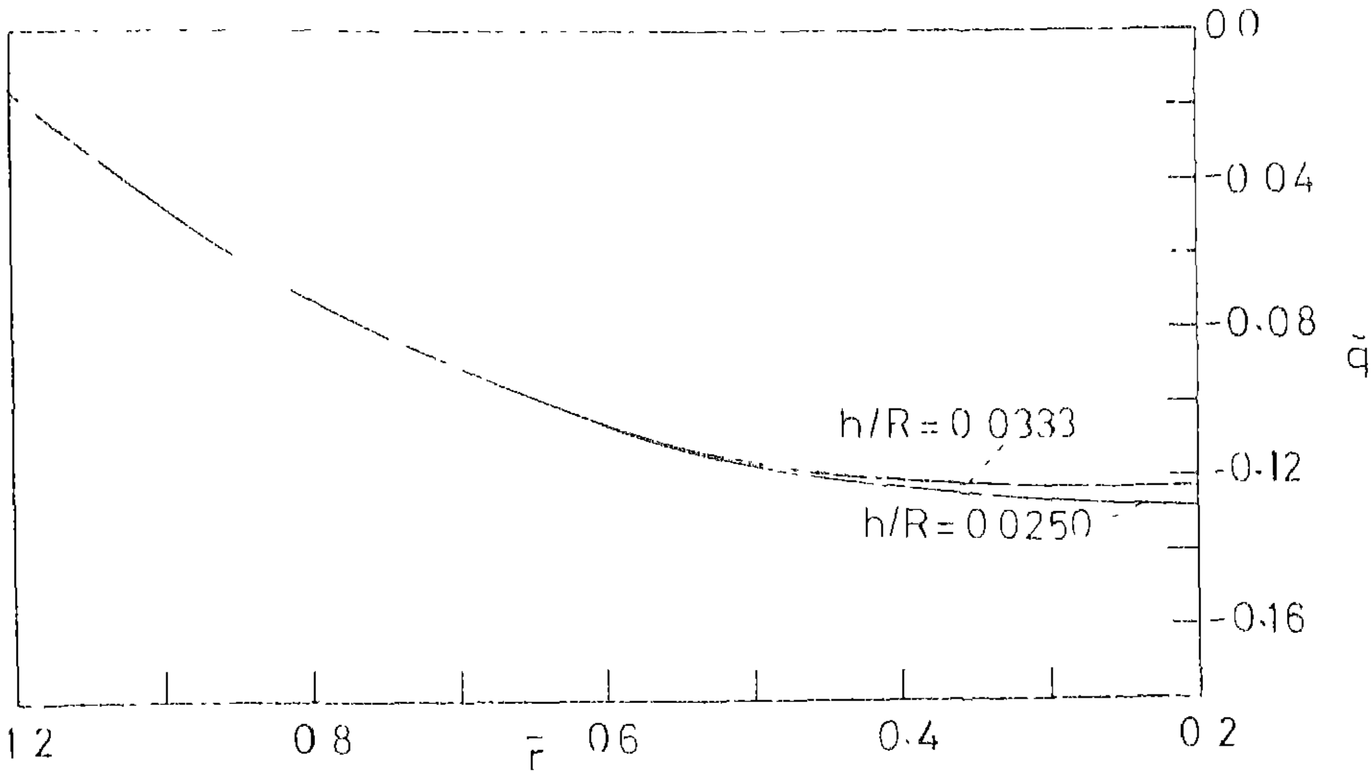


FIG. 4.1a SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (h/R) VALUES

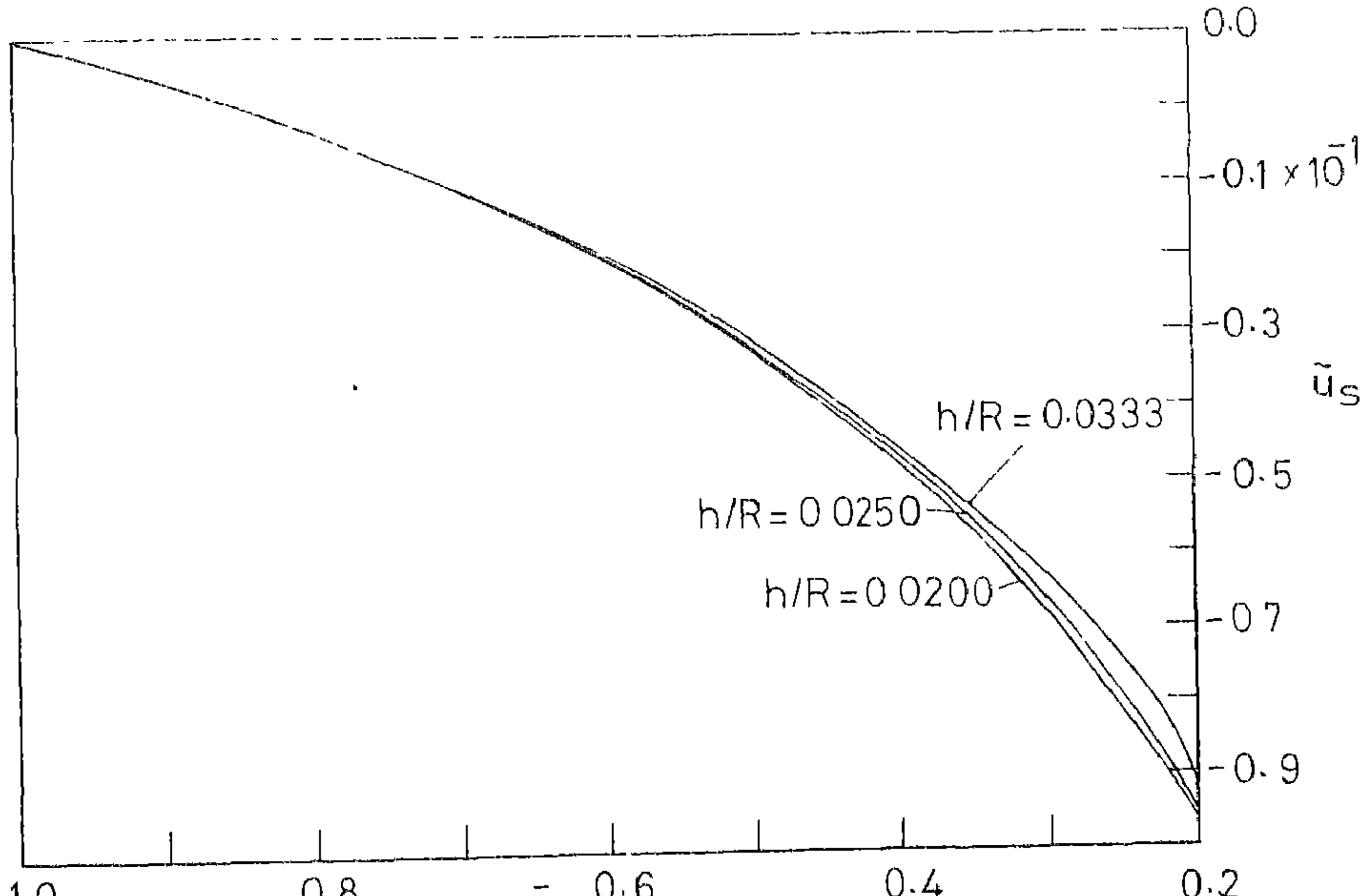


FIG. 4.2a FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{u}_s$ FOR DIFFERENT (h/R) VALUES

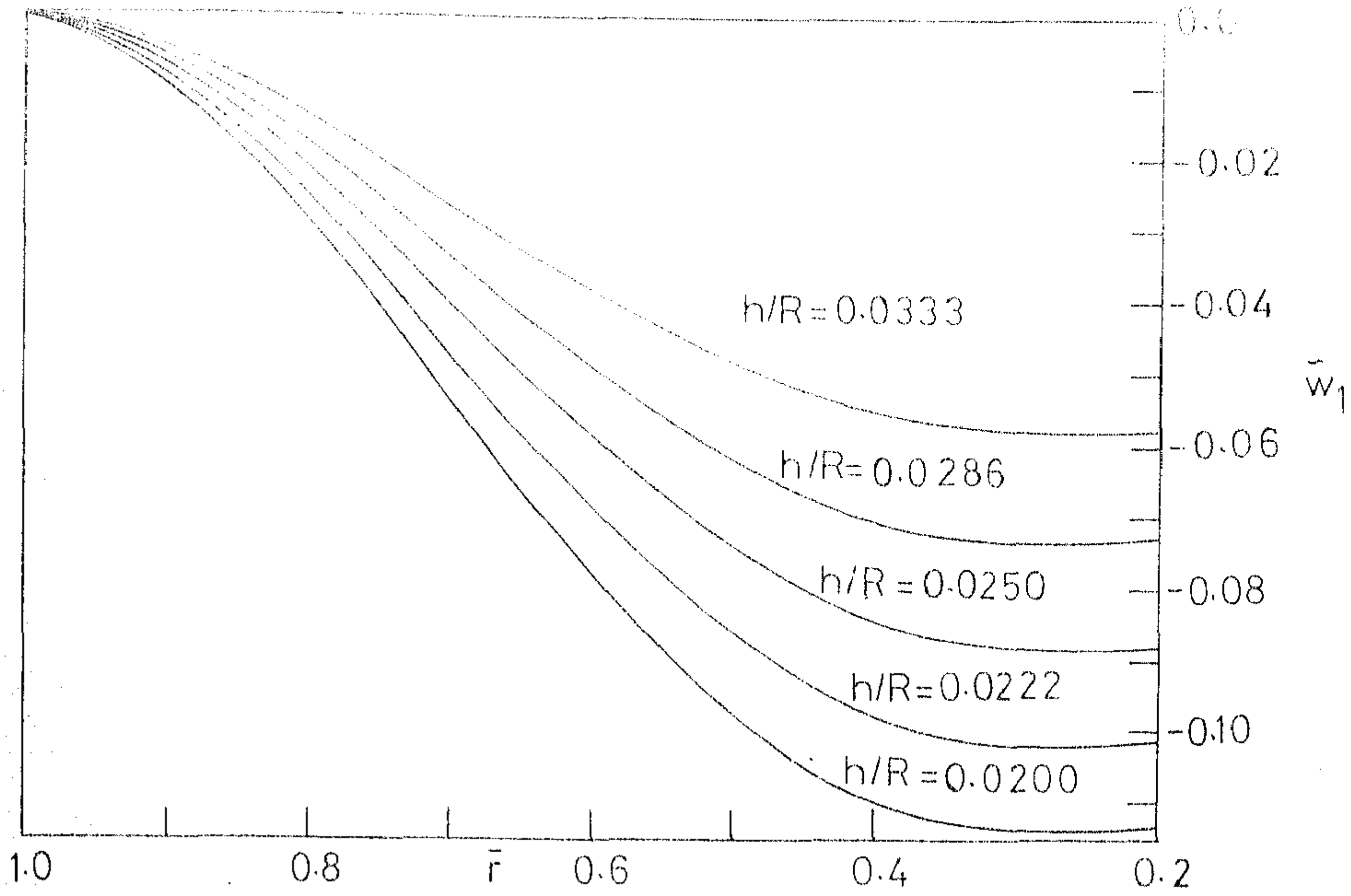


FIG. 4.2b FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (h/R) VALUES

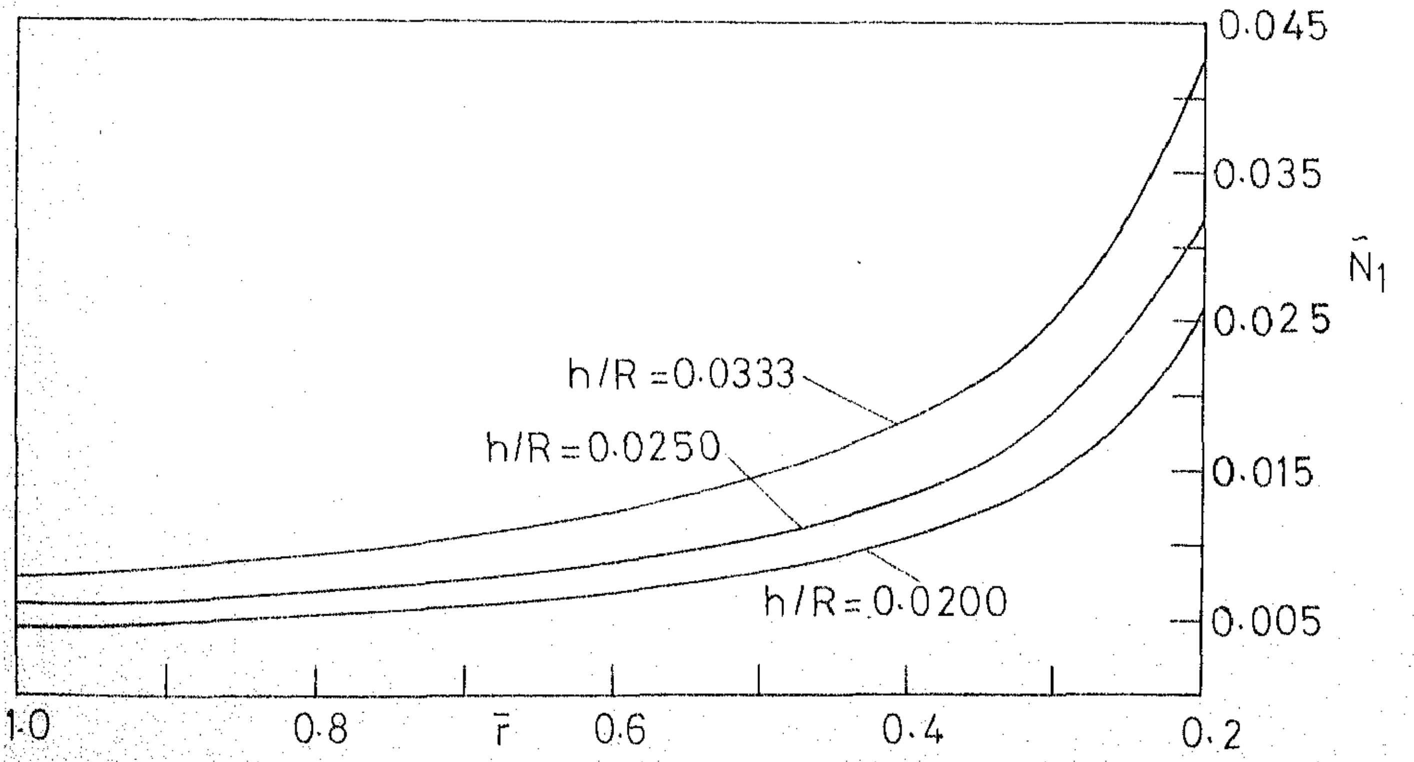


FIG. 4.2c FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{N}_1$ FOR DIFFERENT (h/R) VALUES

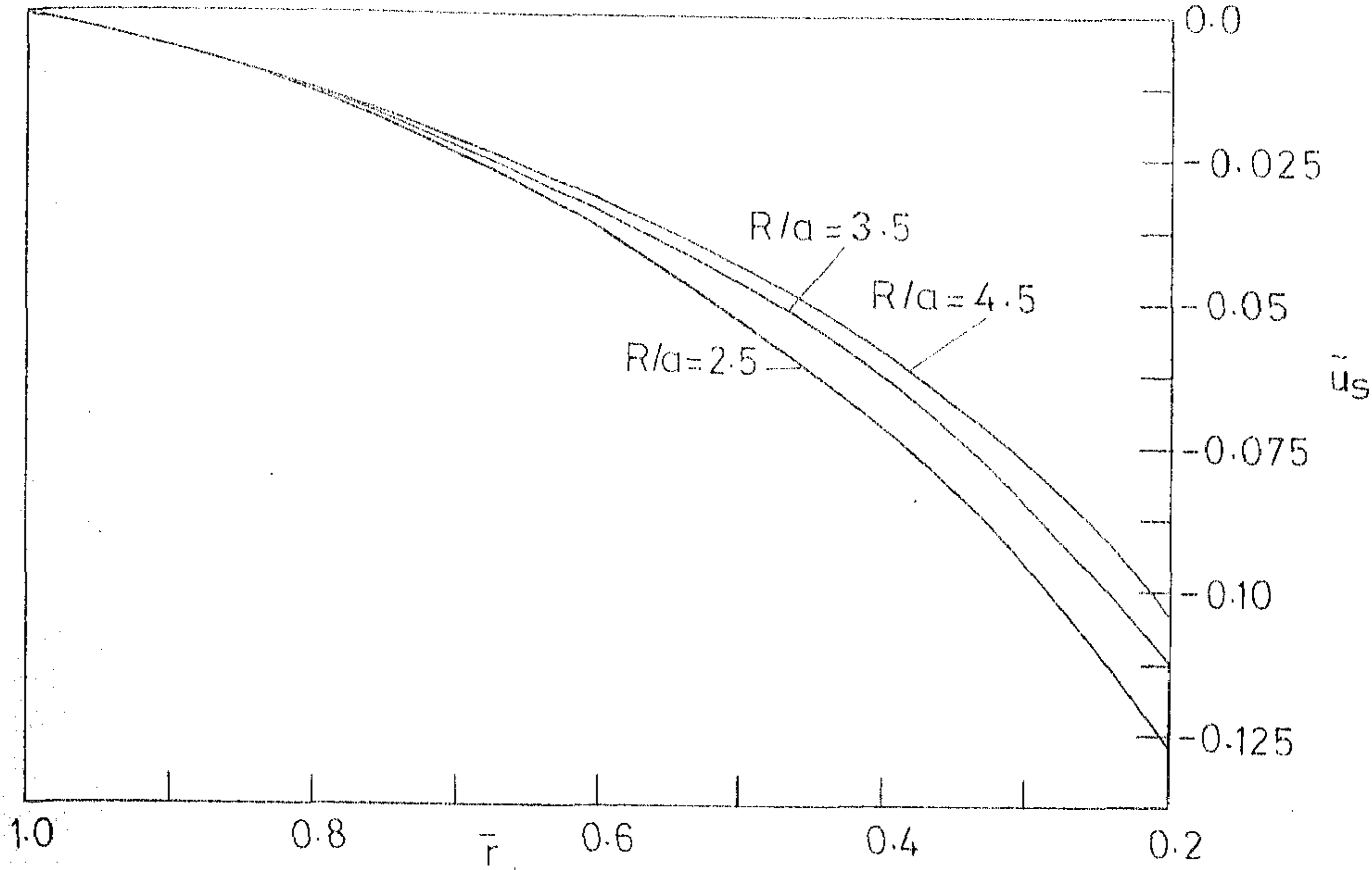


FIG. 4.3a SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\tilde{u}_s$ FOR DIFFERENT (R/a) VALUES

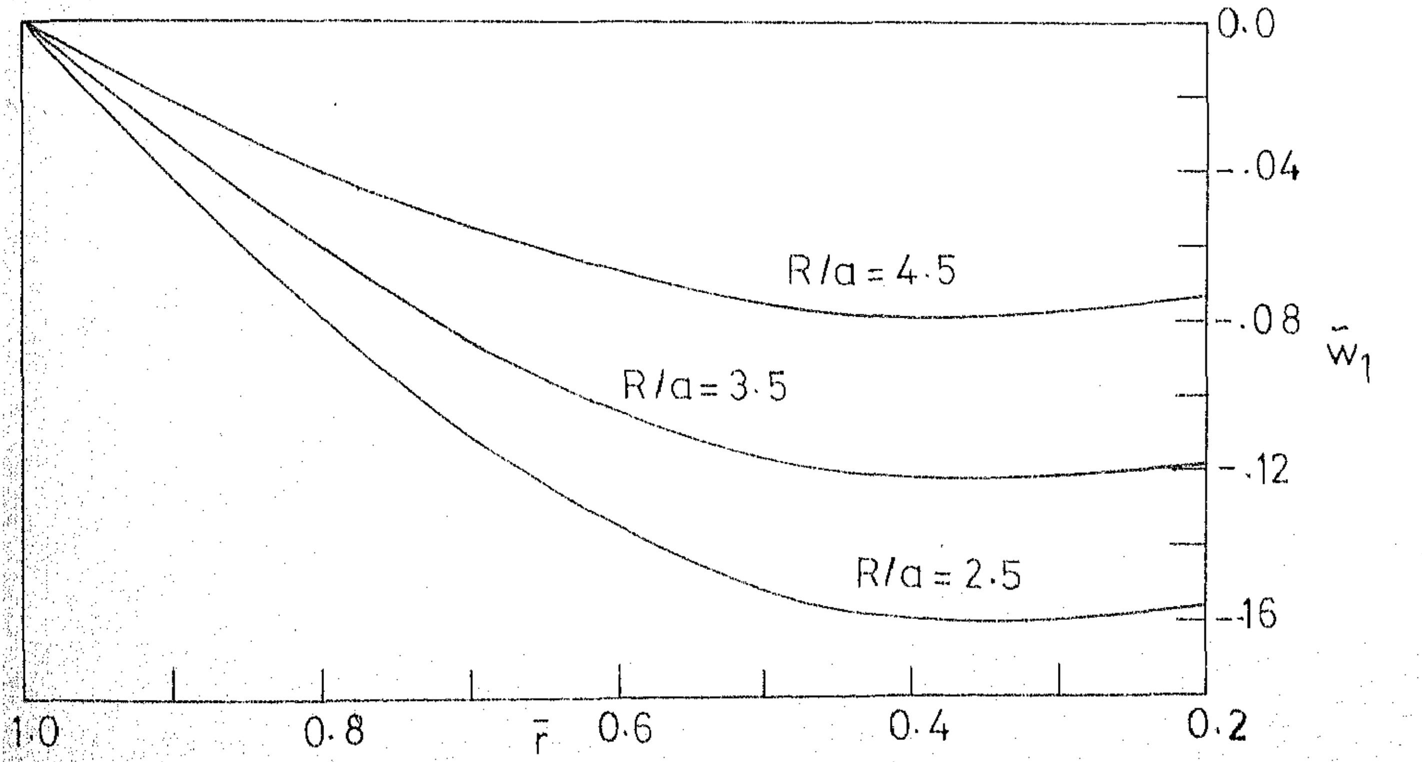


FIG. 4.3b SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (R/a) VALUES

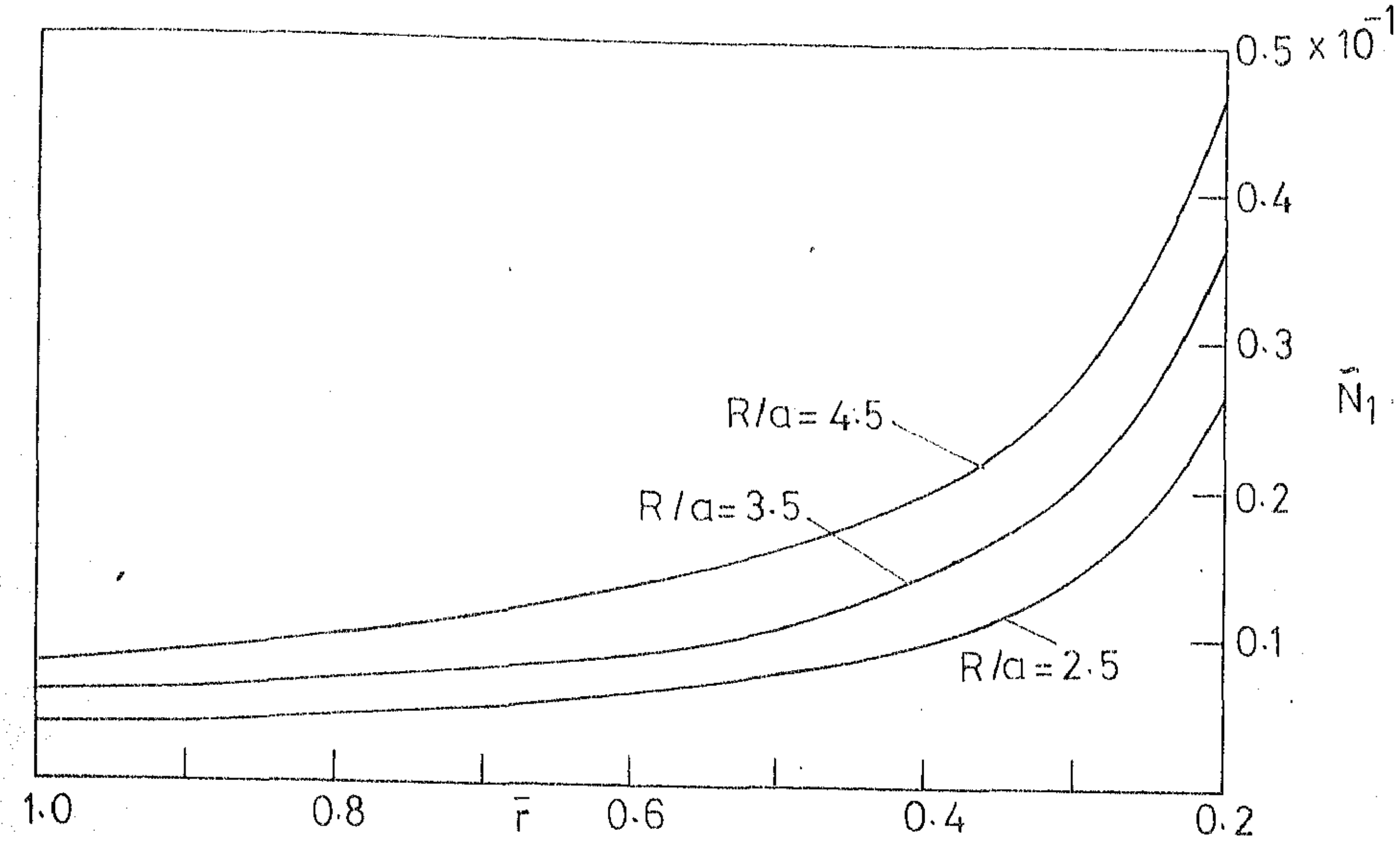


FIG. 4.3c SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\tilde{N}_1$ FOR DIFFERENT (R/a) VALUES

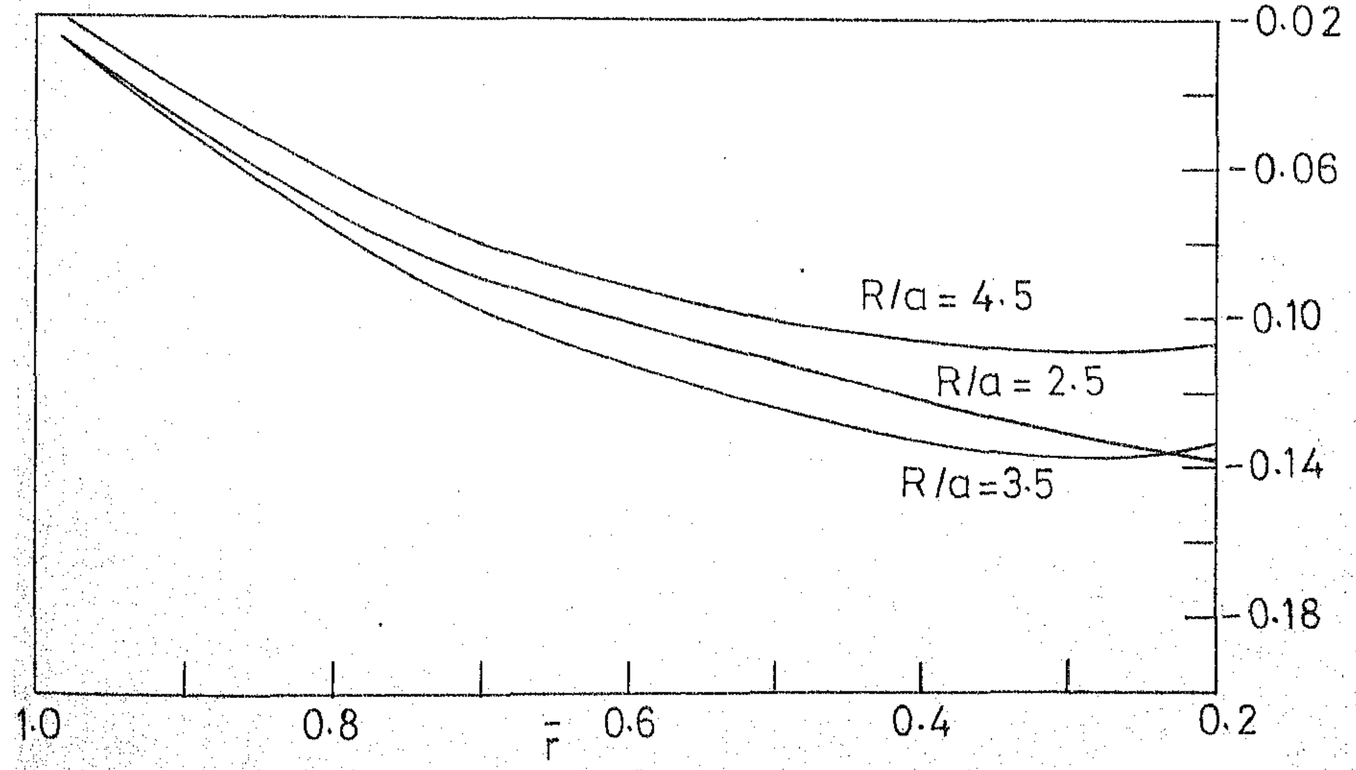


FIG. 4.3d SIMPLY SUPPORTED SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\bar{q}$ FOR DIFFERENT (R/a) VALUES

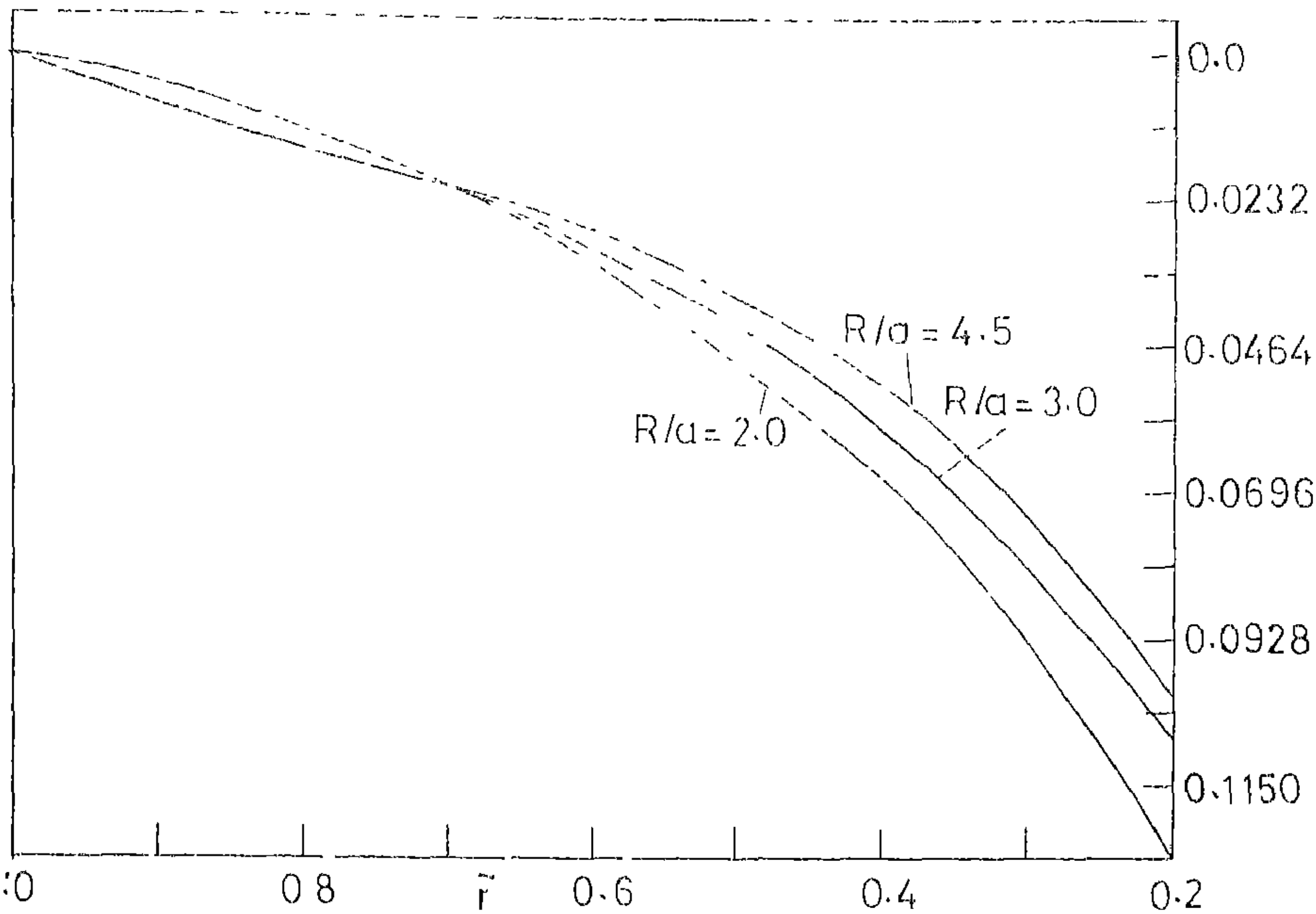


FIG. 4.4a FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\bar{u}_s$ FOR DIFFERENT (R/a) VALUES

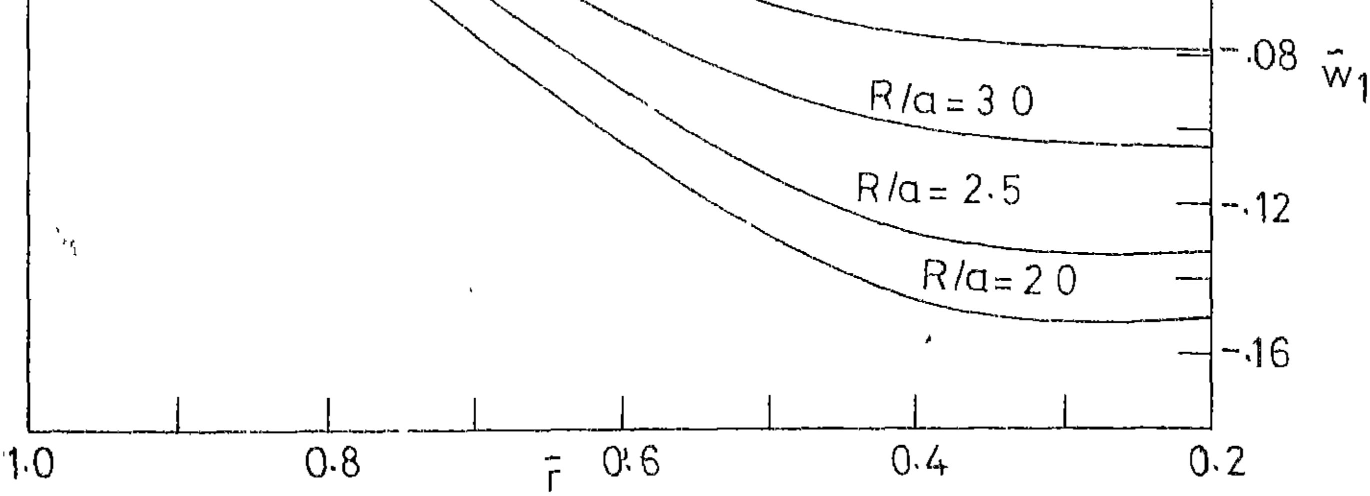


FIG. 4.4b FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\bar{w}_1$ FOR DIFFERENT (R/a) VALUES

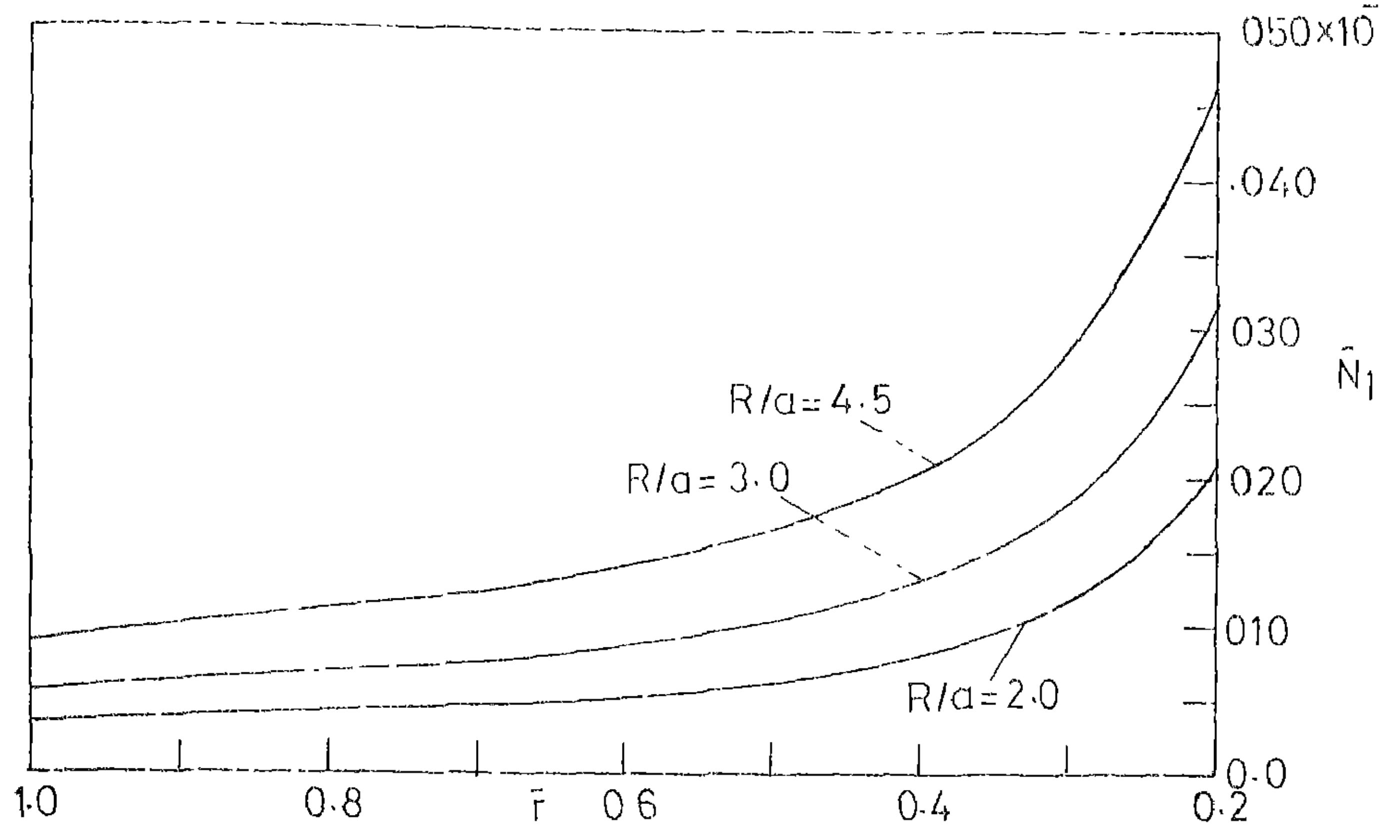


FIG. 4.4c FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\bar{N}_1$ FOR DIFFERENT (R/a) VALUES

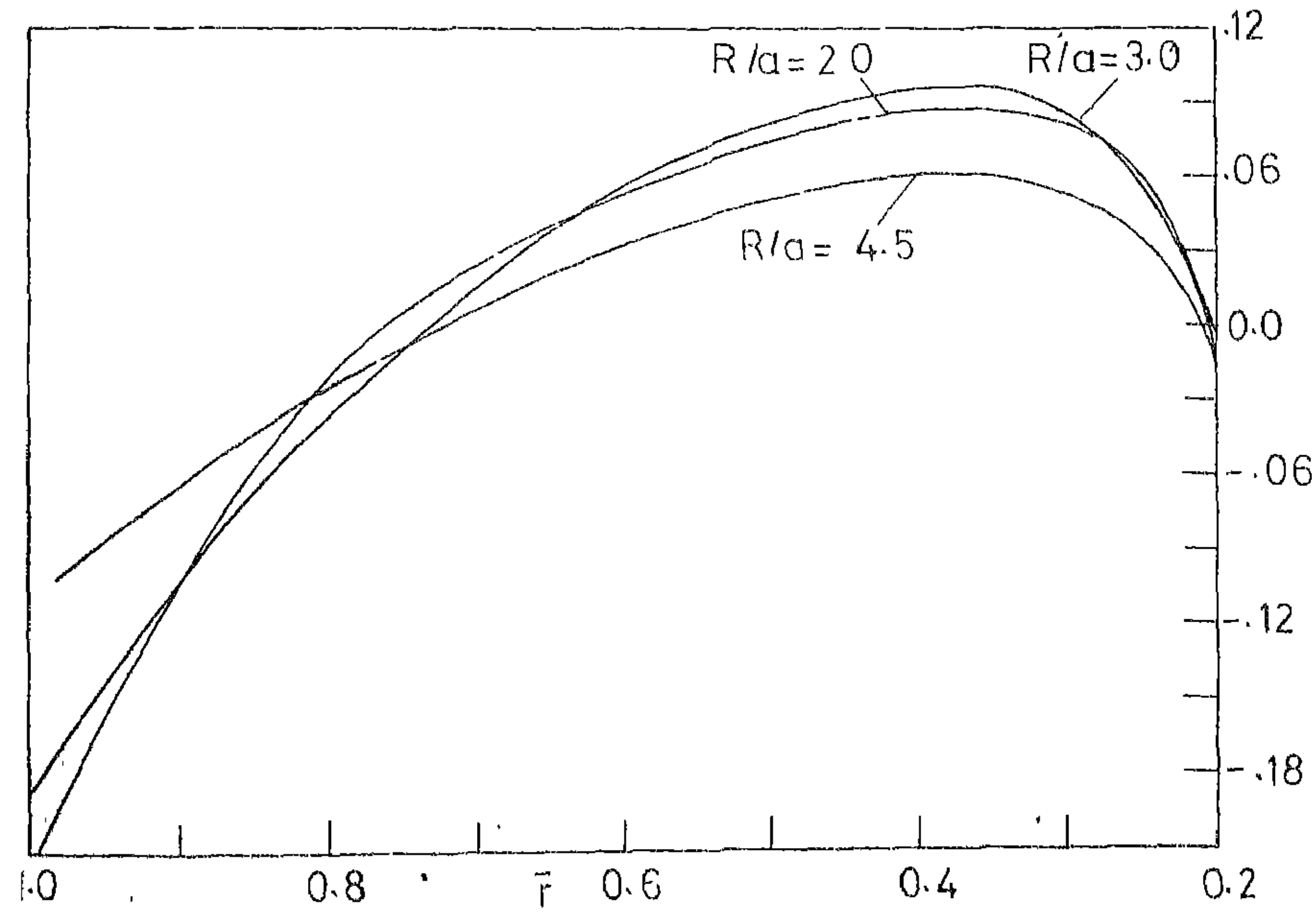


FIG. 4.4d FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\bar{M}_1$ FOR DIFFERENT (R/a) VALUES

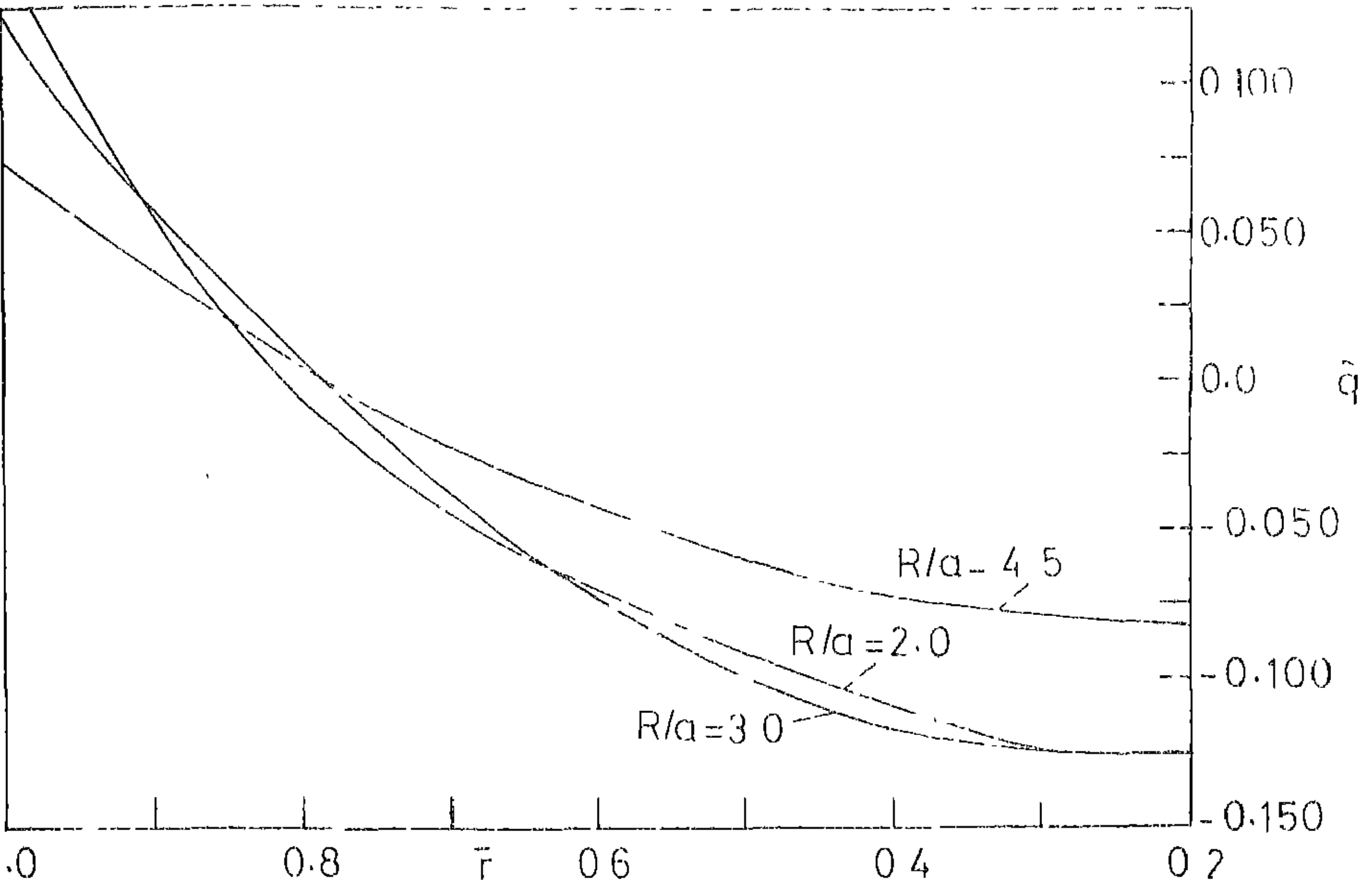


FIG. 4.4c FIXED SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (R/a) VALUES

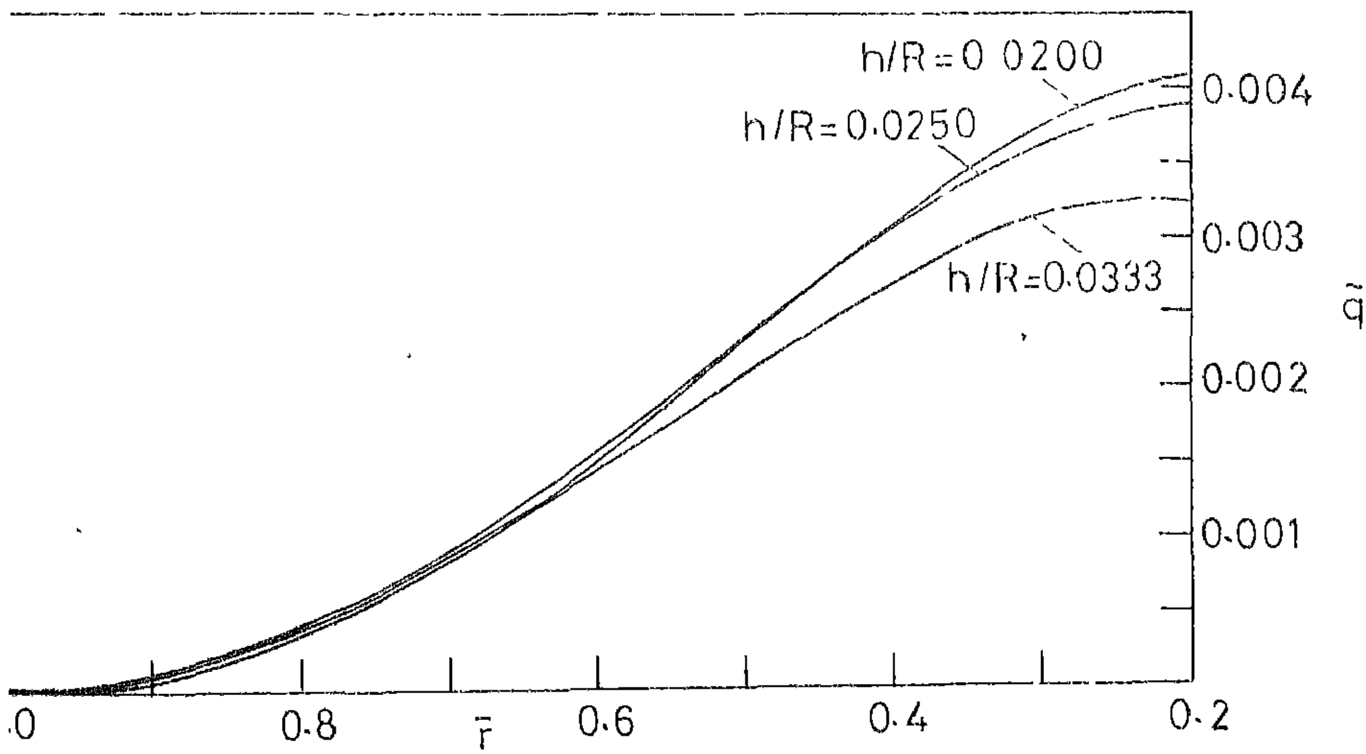
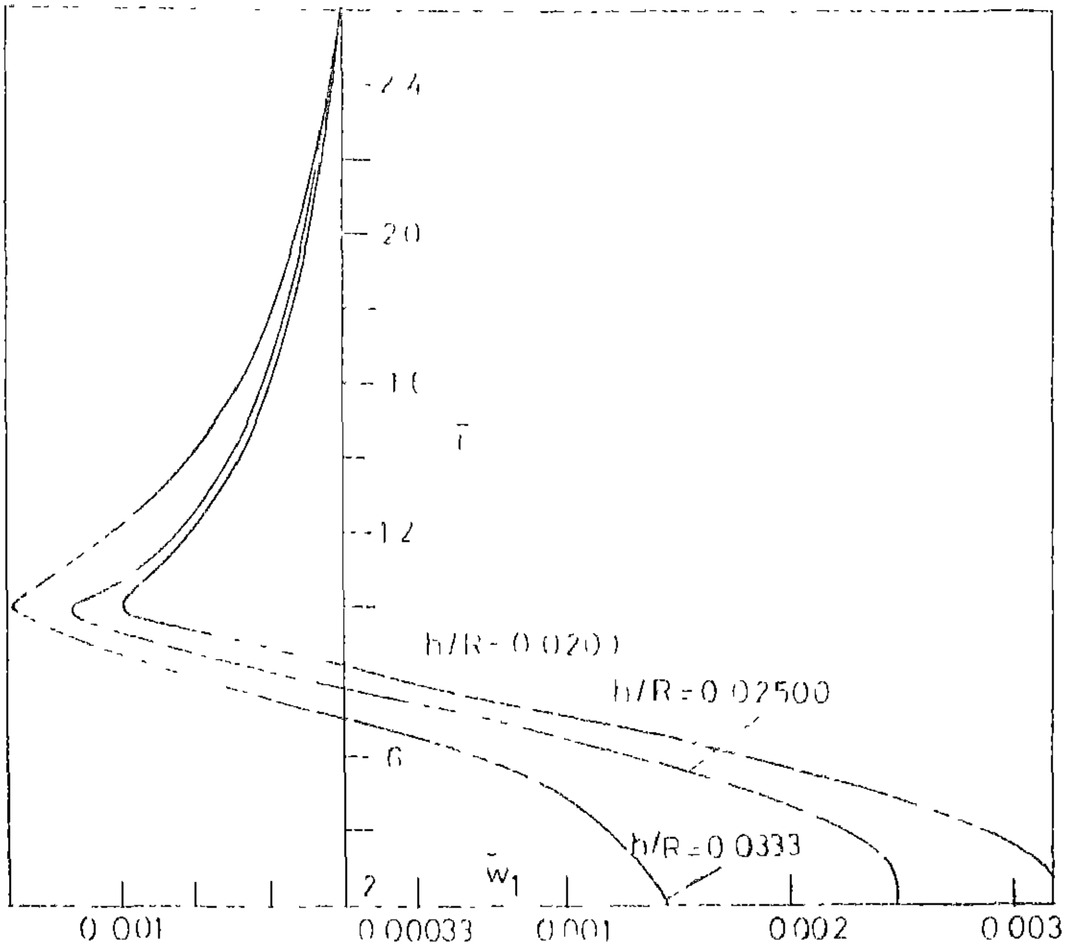


FIG. 4.5a FREE SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (h/R) VALUES



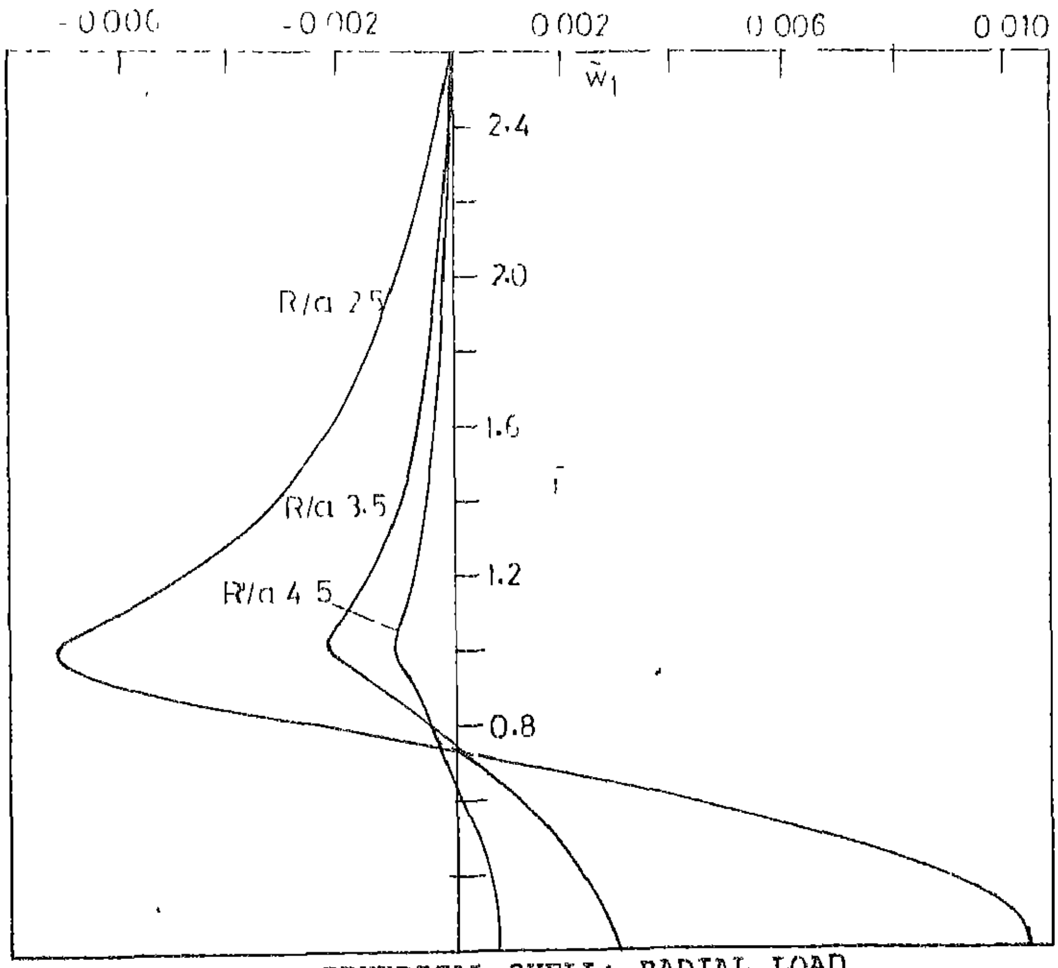


FIG. 4.6a FREE SPHERICAL SHELL; RADIAL LOAD VARIATION OF $\bar{w}_1$ FOR DIFFERENT (R/a) VALUES

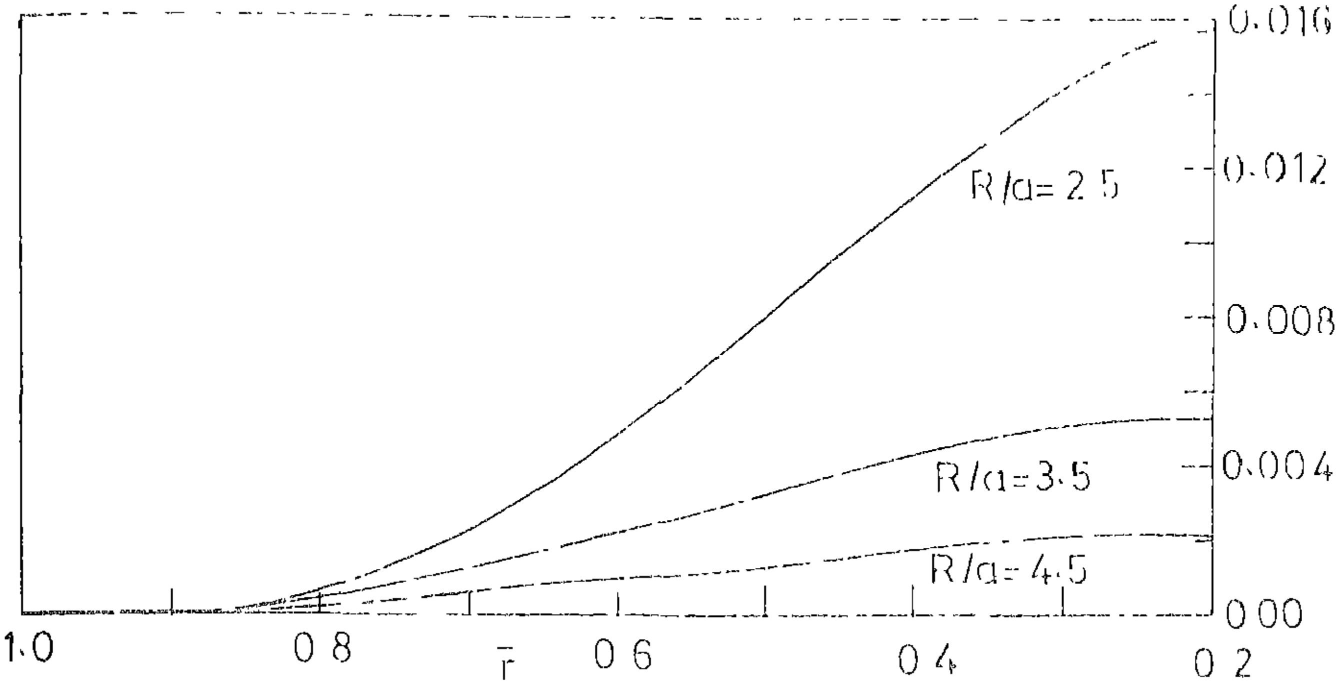


FIG. 4.6b FREE SPHERICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{q}$ FOR DIFFERENT (R/a) VALUES

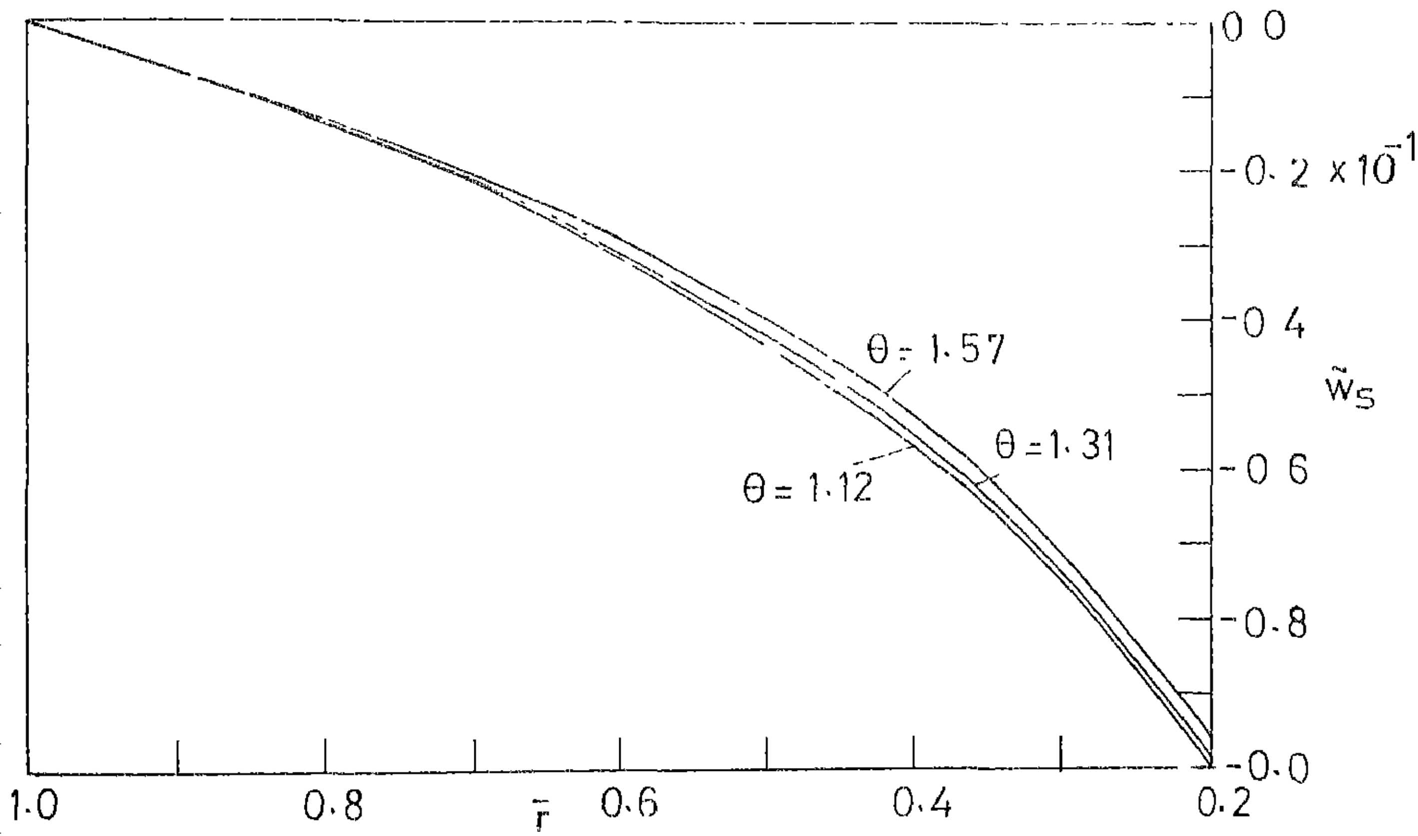
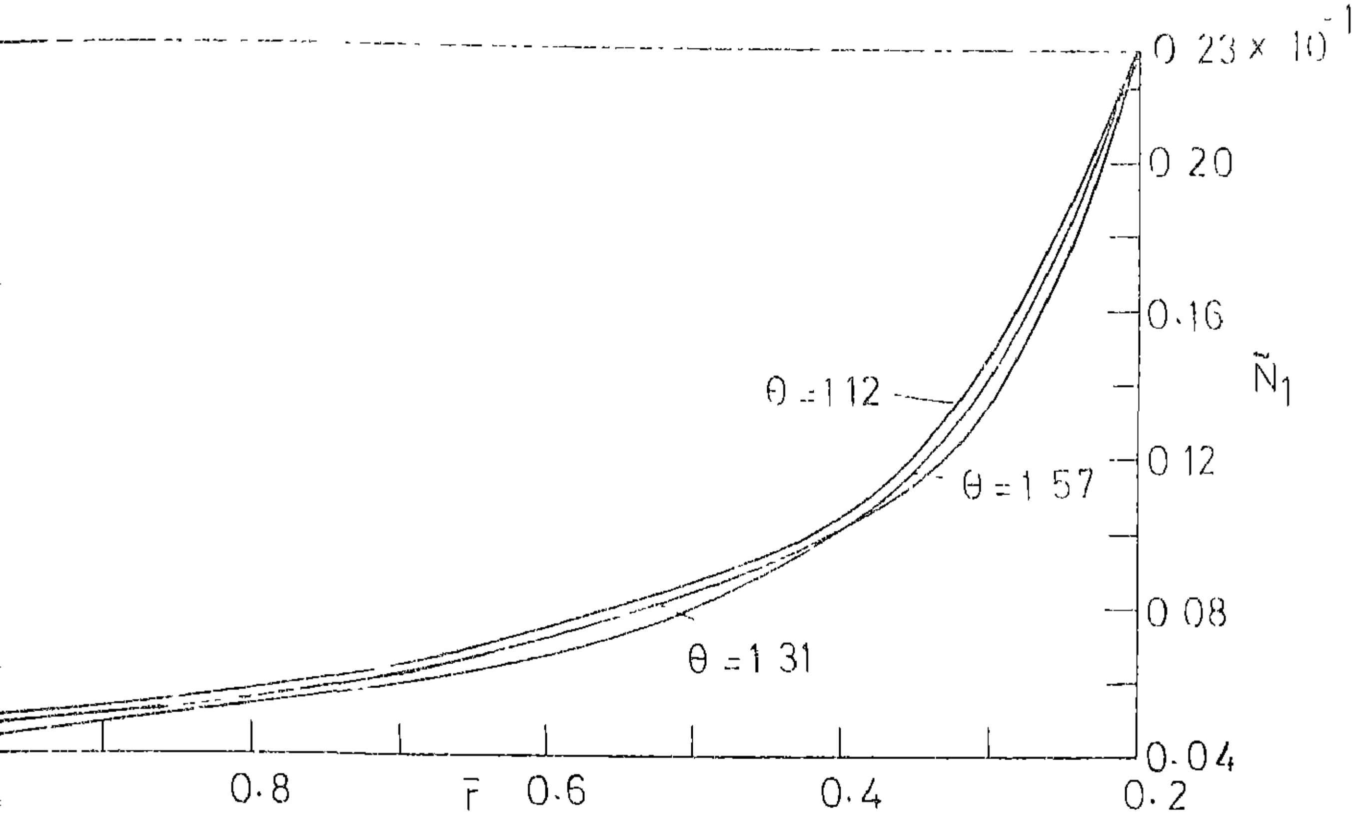
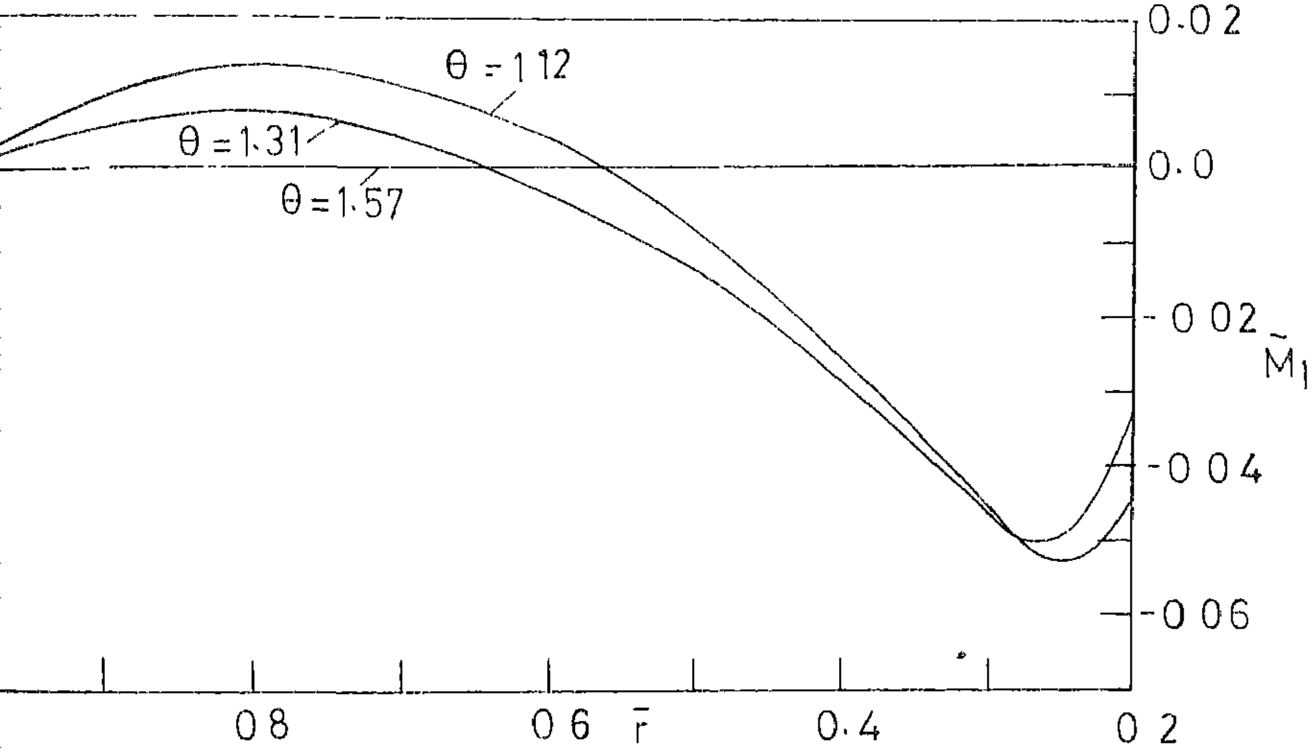


FIG. 4.7a SIMPLY SUPPORTED CONICAL SHELL; RADIAL LOAD
VARIATION OF $\tilde{w}_s$ FOR DIFFERENT θ VALUES



G. 4.7b SIMPLY SUPPORTED CONICAL SHELL; RADIAL LOAD VARIATION OF $\tilde{N}_1$ FOR DIFFERENT θ VALUES



G. 4.7c SIMPLY SUPPORTED CONICAL SHELL; RADIAL LOAD VARIATION OF $\tilde{M}_1$ FOR DIFFERENT θ VALUES

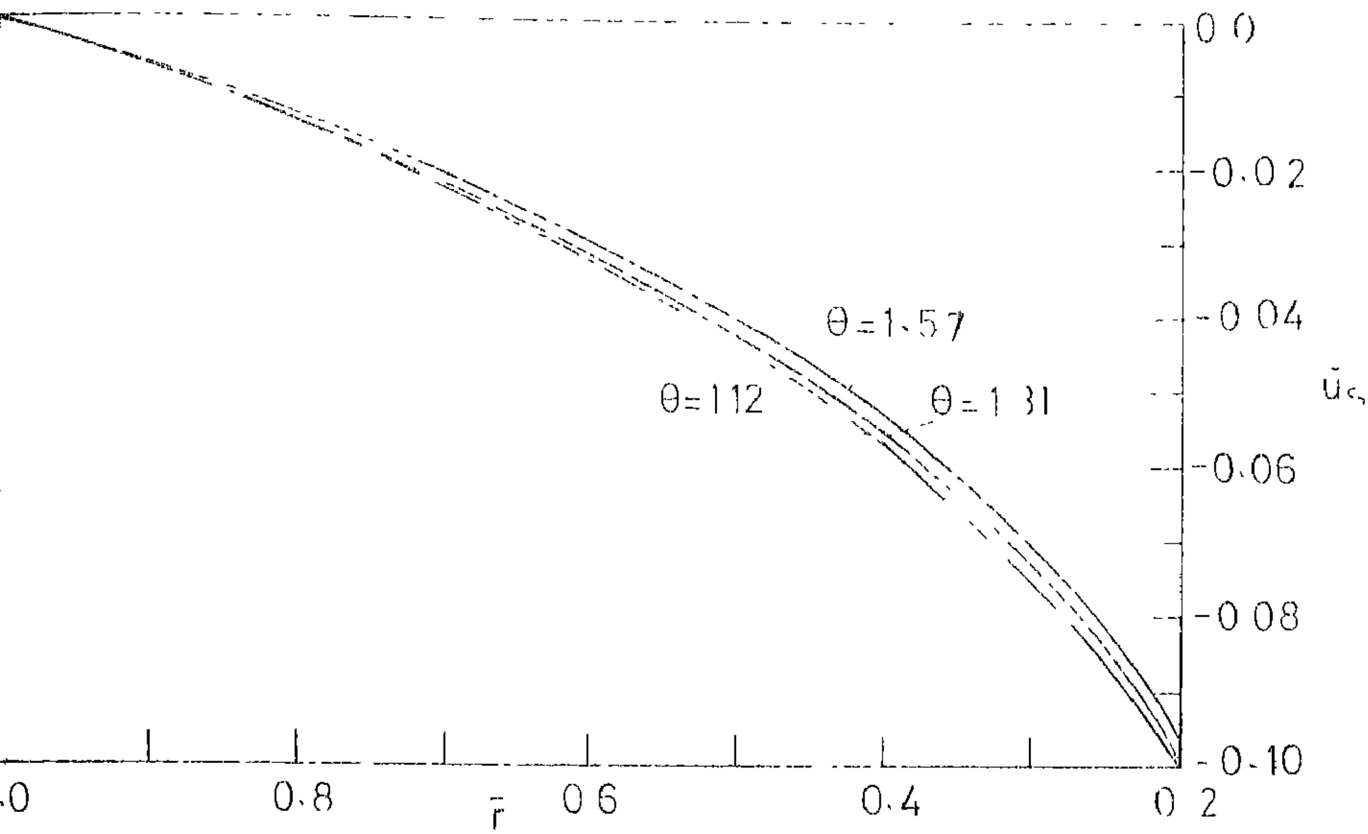


FIG. 4.8a FIXED CONICAL SHELL POINT LOAD VARIATION OF $\bar{u}_s$ FOR DIFFERENT θ VALUES

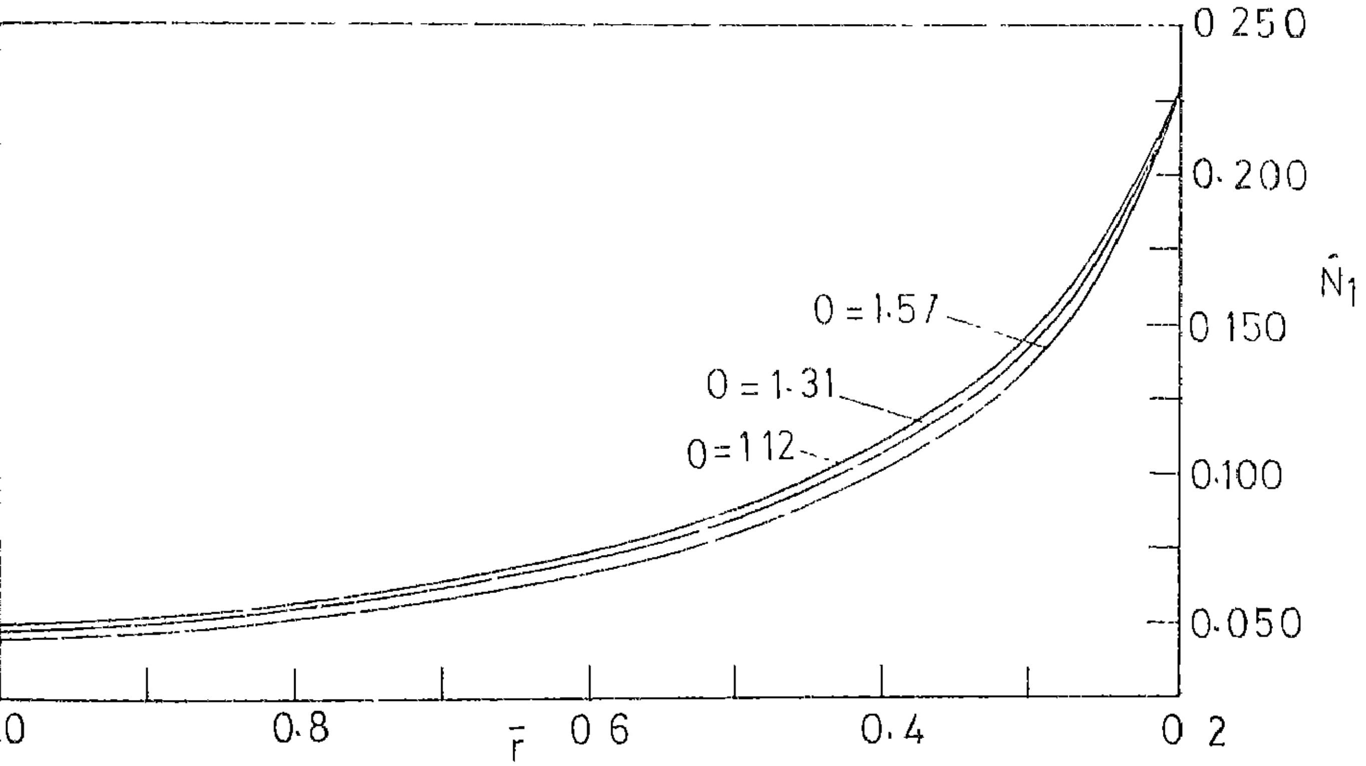


FIG. 4.8b FIXED CONICAL SHELL POINT LOAD VARIATION OF $\hat{N}_1$ FOR DIFFERENT θ VALUES

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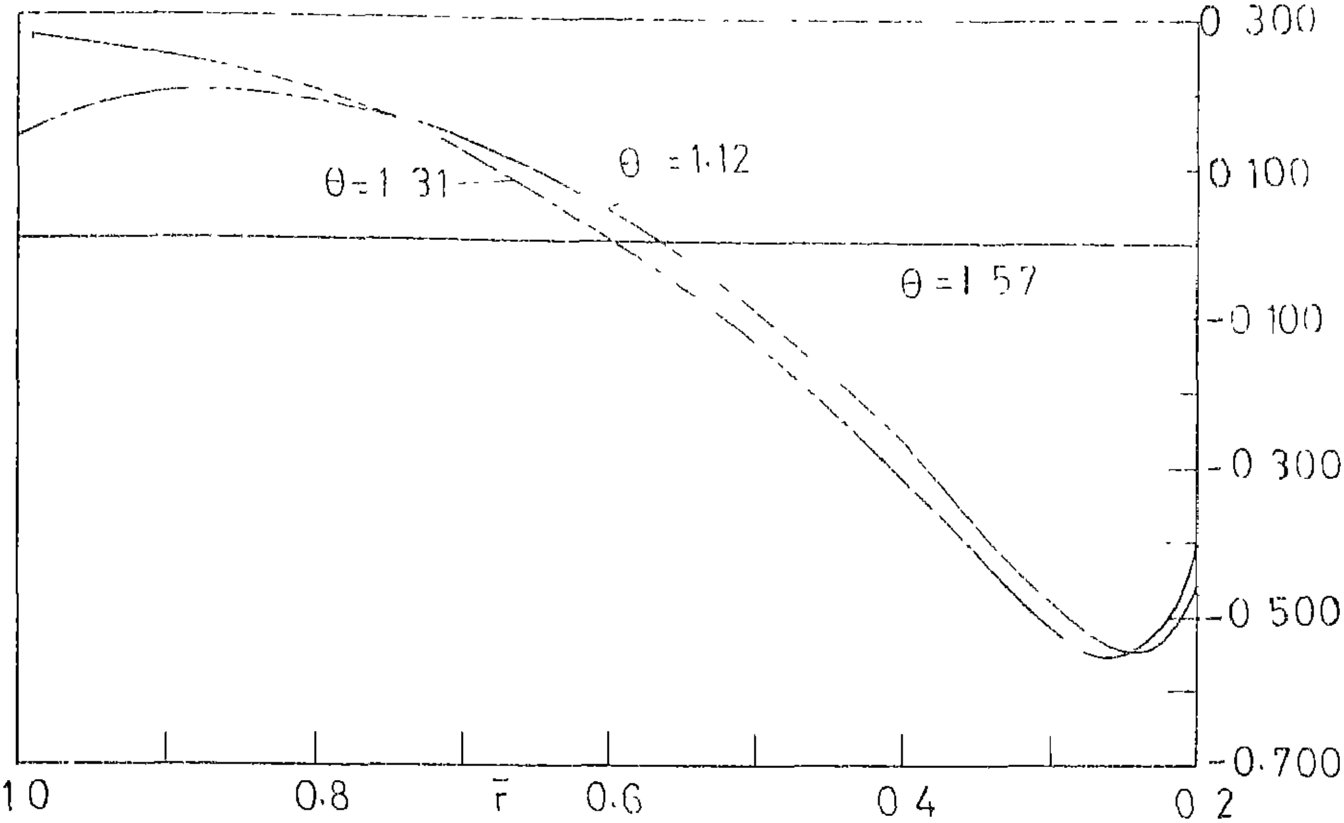

FIG. 4.8c FIXED, CONICAL SHELL; RADIAL LOAD
VARIATION OF $\bar{M}_1$ FOR DIFFERENT θ VALUES

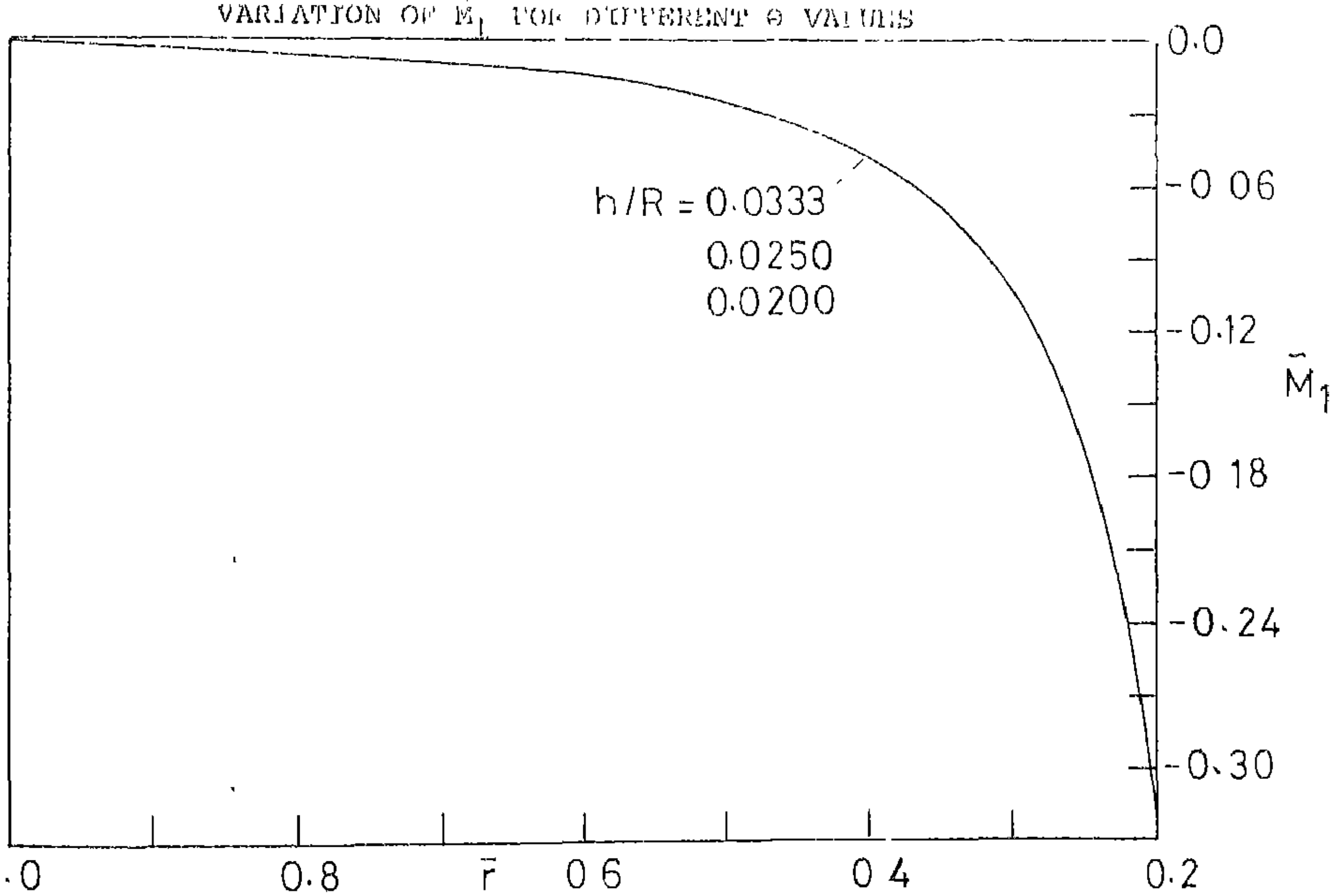

FIG. 4.9 SIMPLY SUPPORTED SPHERICAL SHELL; MOMENT
VARIATION OF $\bar{M}_1$ FOR DIFFERENT (h/R) VALUES

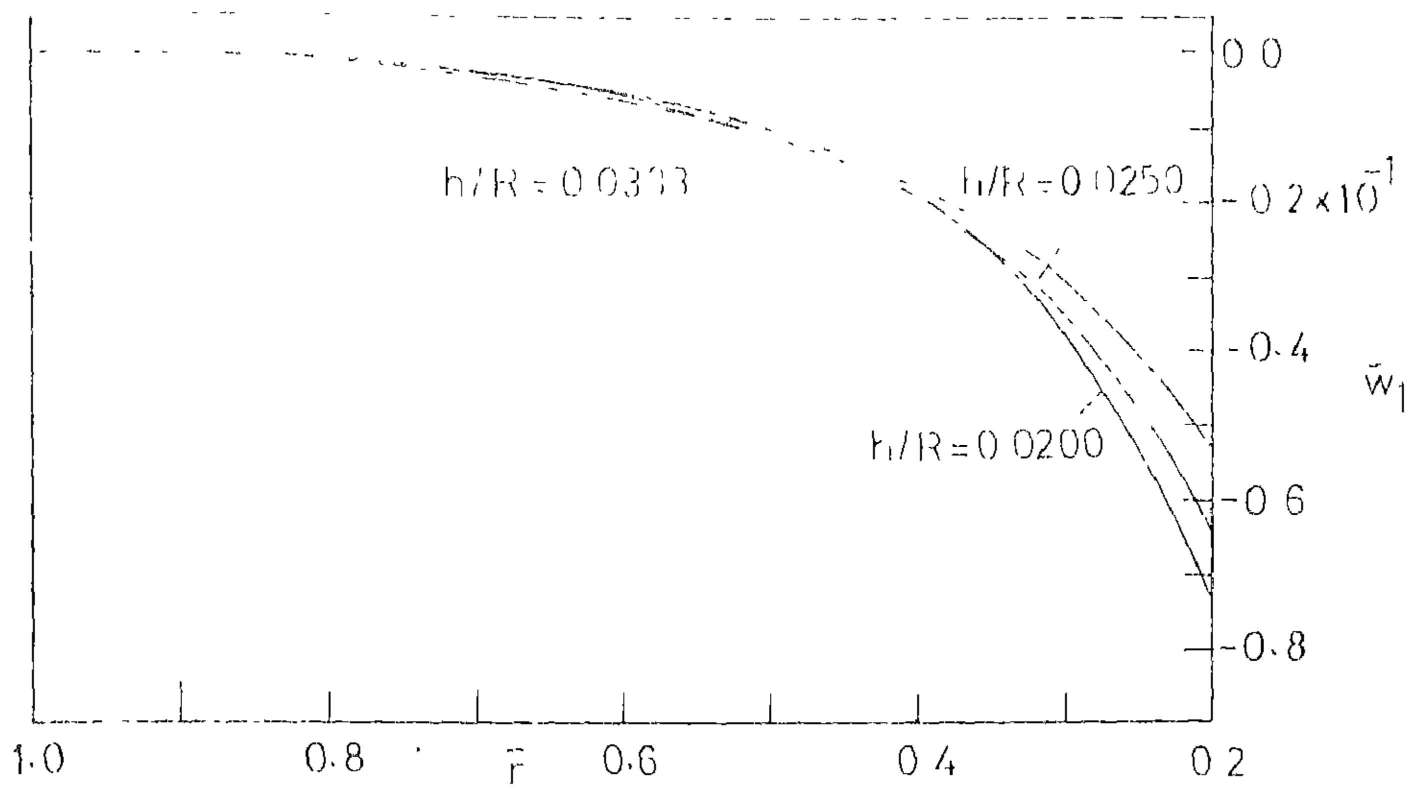

FIG. 4.10a FIXED SPHERICAL SHELL; MOMENT
VARIATION OF W FOR DIFFERENT (h/R) VALUES

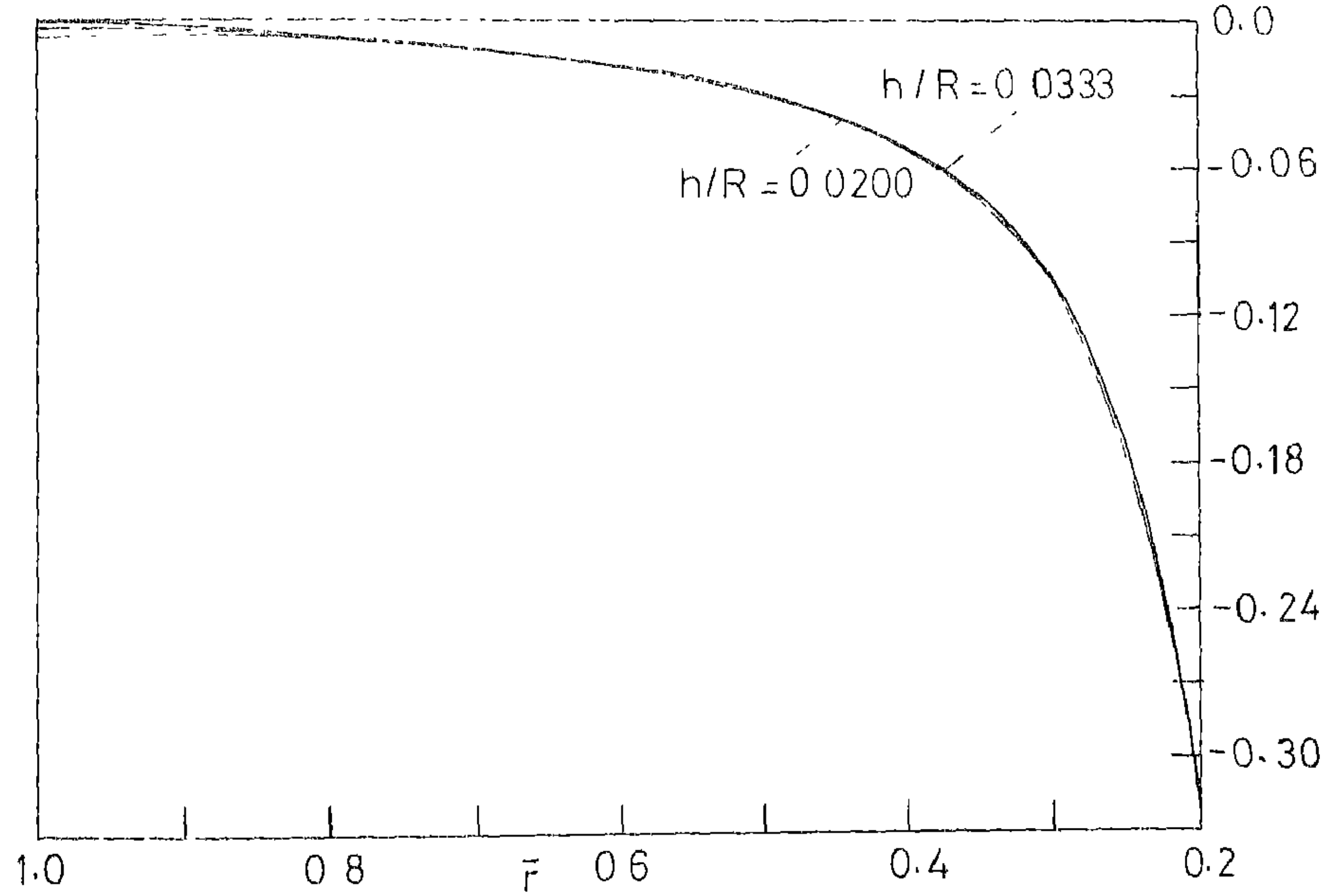

FIG. 4.10b FIXED SPHERICAL SHELL; MOMENT
VARIATION OF $\tilde{M}_1$ FOR DIFFERENT (h/R) VALUES

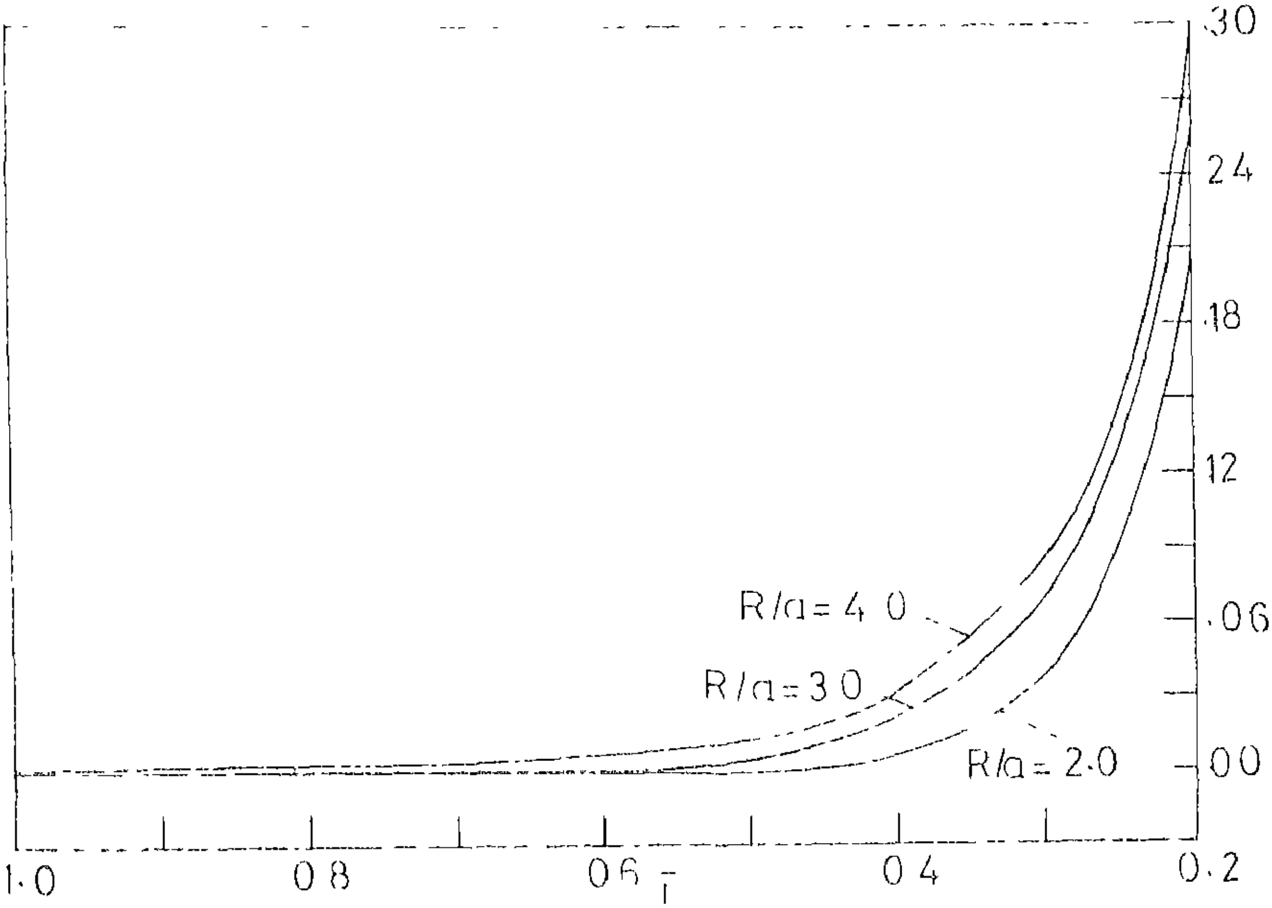

FIG. 4.11 SIMPLY SUPPORTED SPHERICAL SHELL; MOMENT VARIATION OF $\bar{M}_r$ FOR DIFFERENT (R/a) VALUES

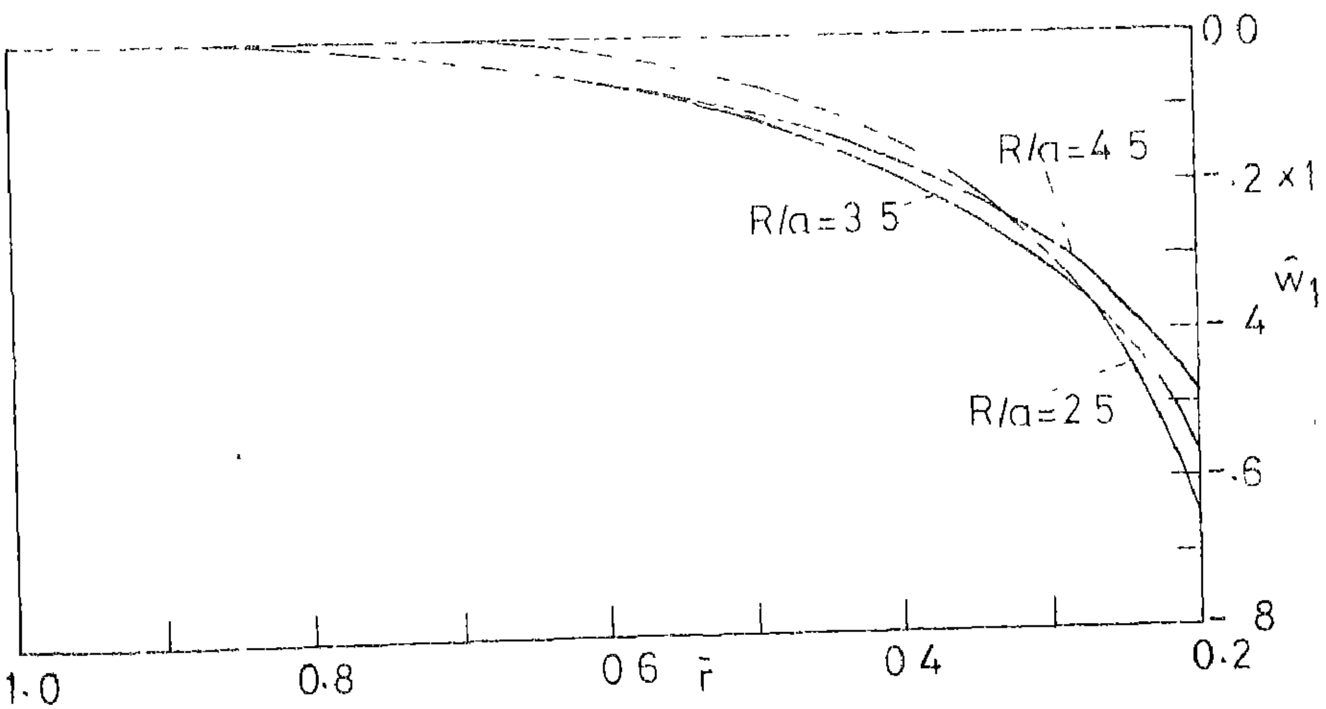

FIG. 4.12a FIXED SPHERICAL SHELL; MOMENT VARIATION OF $\bar{w}_1$ FOR DIFFERENT (R/a) VALUES

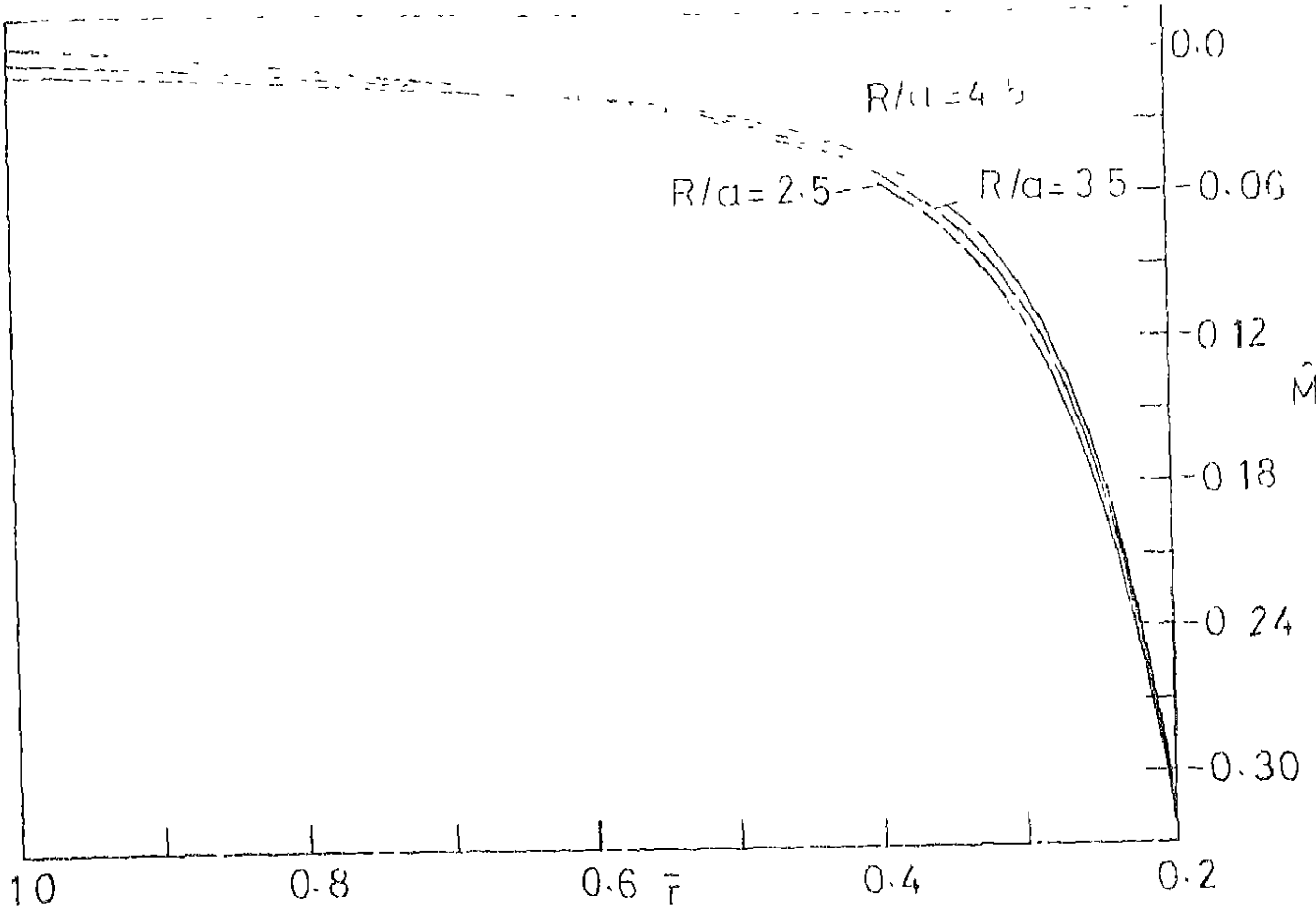

FIG. 4.12b FIXED SPHERICAL SHELL; MOMENT
VARIATION OF $\tilde{M}_1$ FOR DIFFERENT (R/a) VALUES

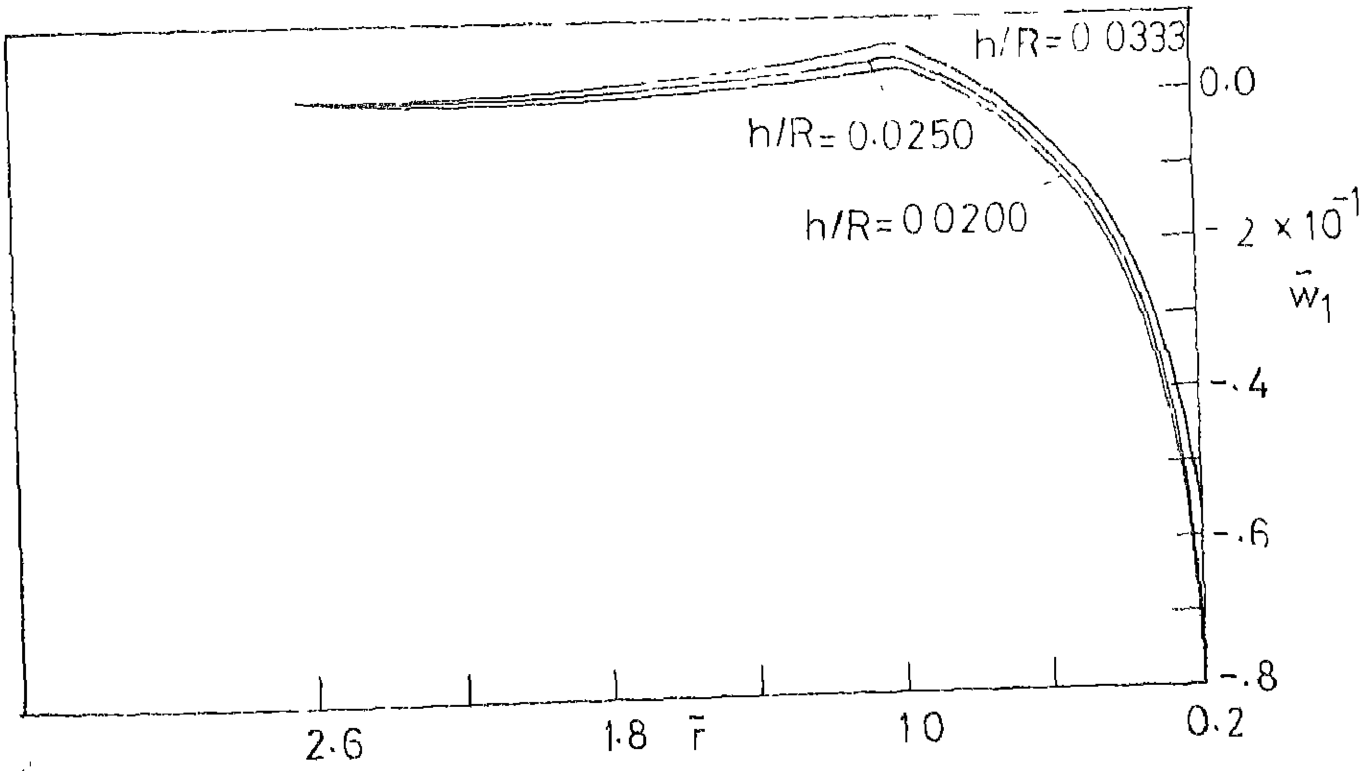

FIG. 4.13a FREE SPHERICAL SHELL; MOMENT
VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (h/R) VALUES

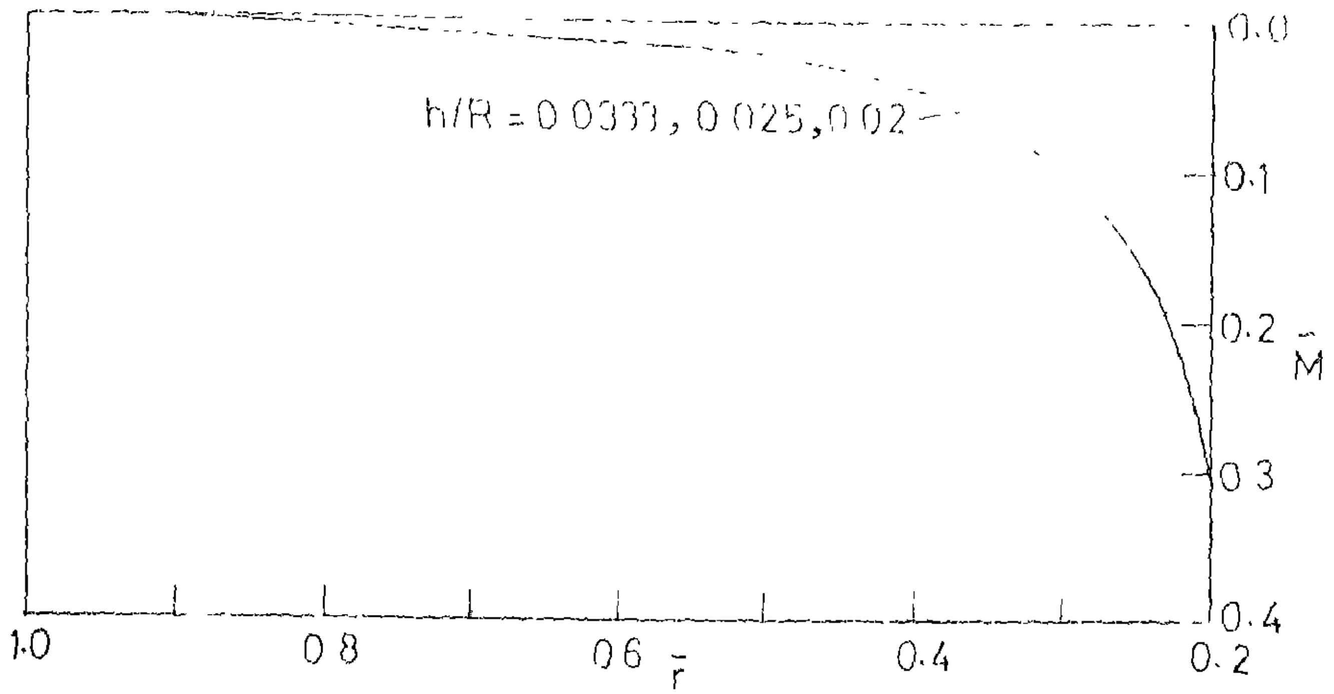

FIG. 4.13b FREE SPHERICAL SHELL; MOMENT VARIATION OF $\bar{M}_1$ FOR DIFFERENT (h/R) VALUES

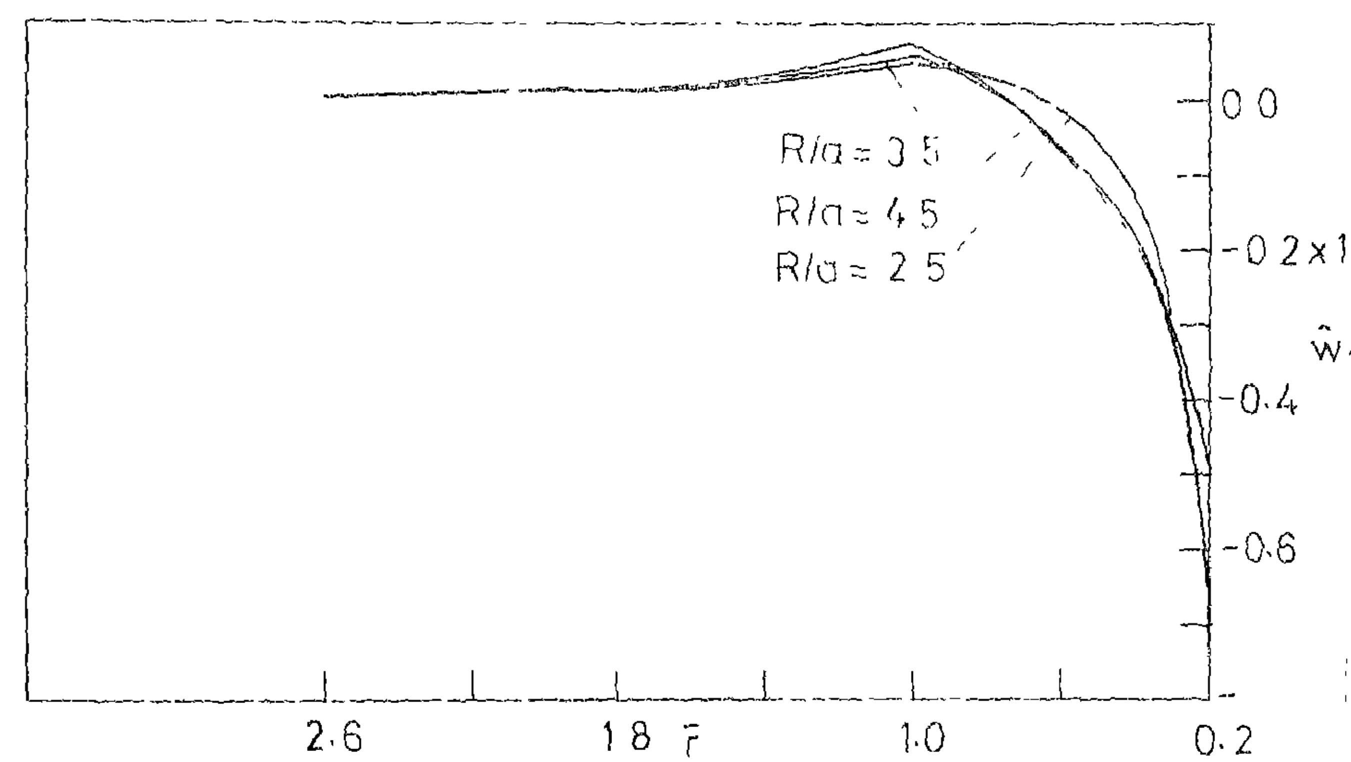

FIG. 4.14a FREE SPHERICAL SHELL; MOMENT VARIATION OF $\tilde{w}_1$ FOR DIFFERENT (R/a) VALUES

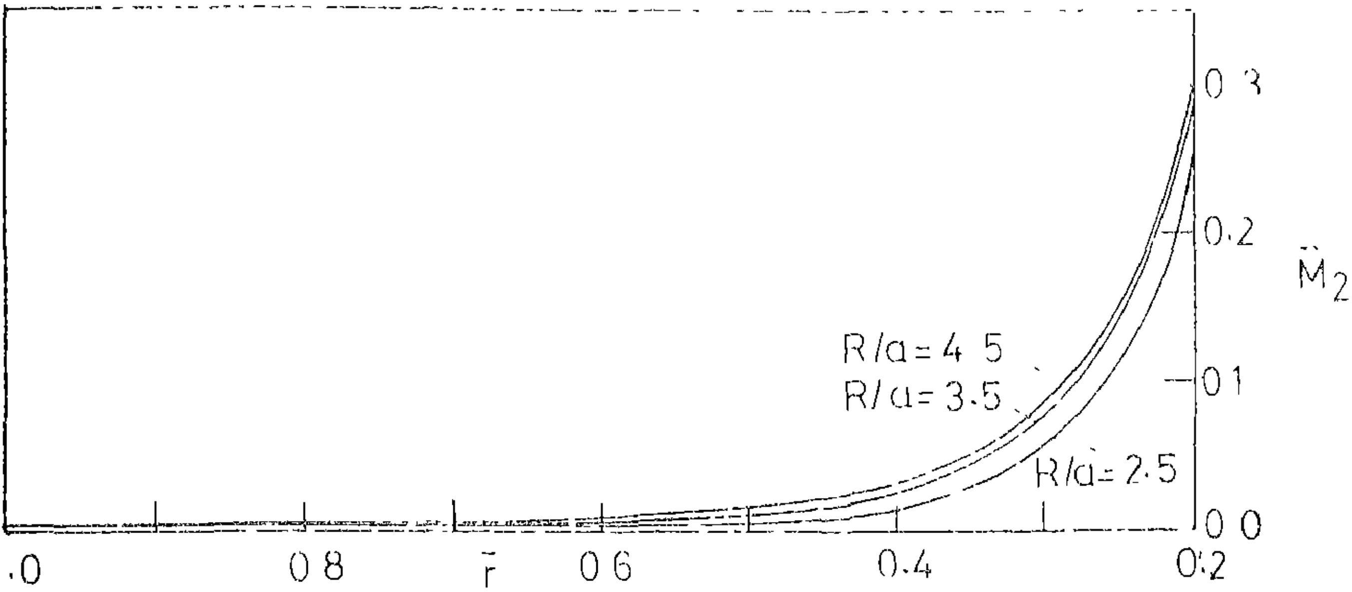

FIG. 4.14b FREE SPHERICAL SHELL; MOMENT VARIATION OF $\bar{M}_2$ FOR DIFFERENT (R/a) VALUES

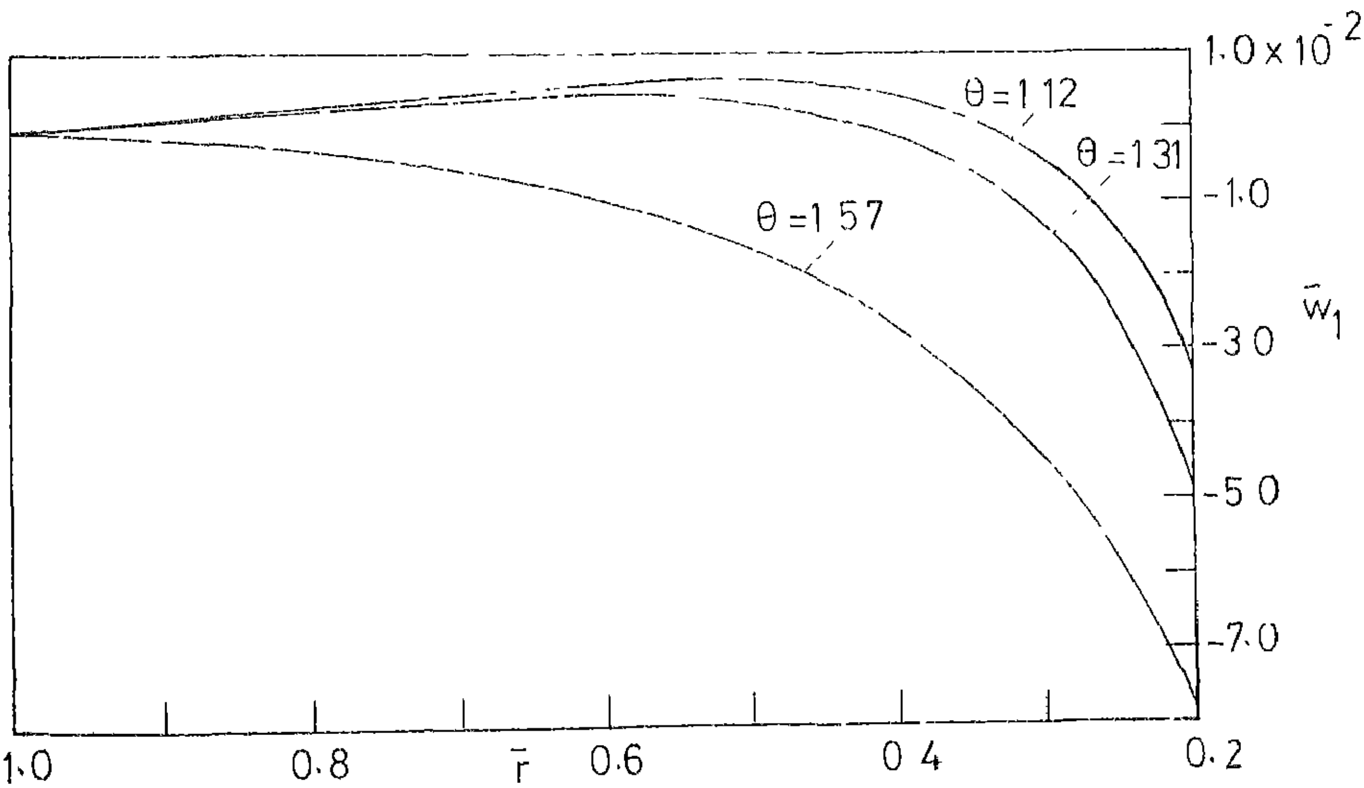

FIG. 4.15a SIMPLY SUPPORTED CONICAL SHELL; MOMENT VARIATION OF $\bar{w}_1$ FOR DIFFERENT θ VALUES

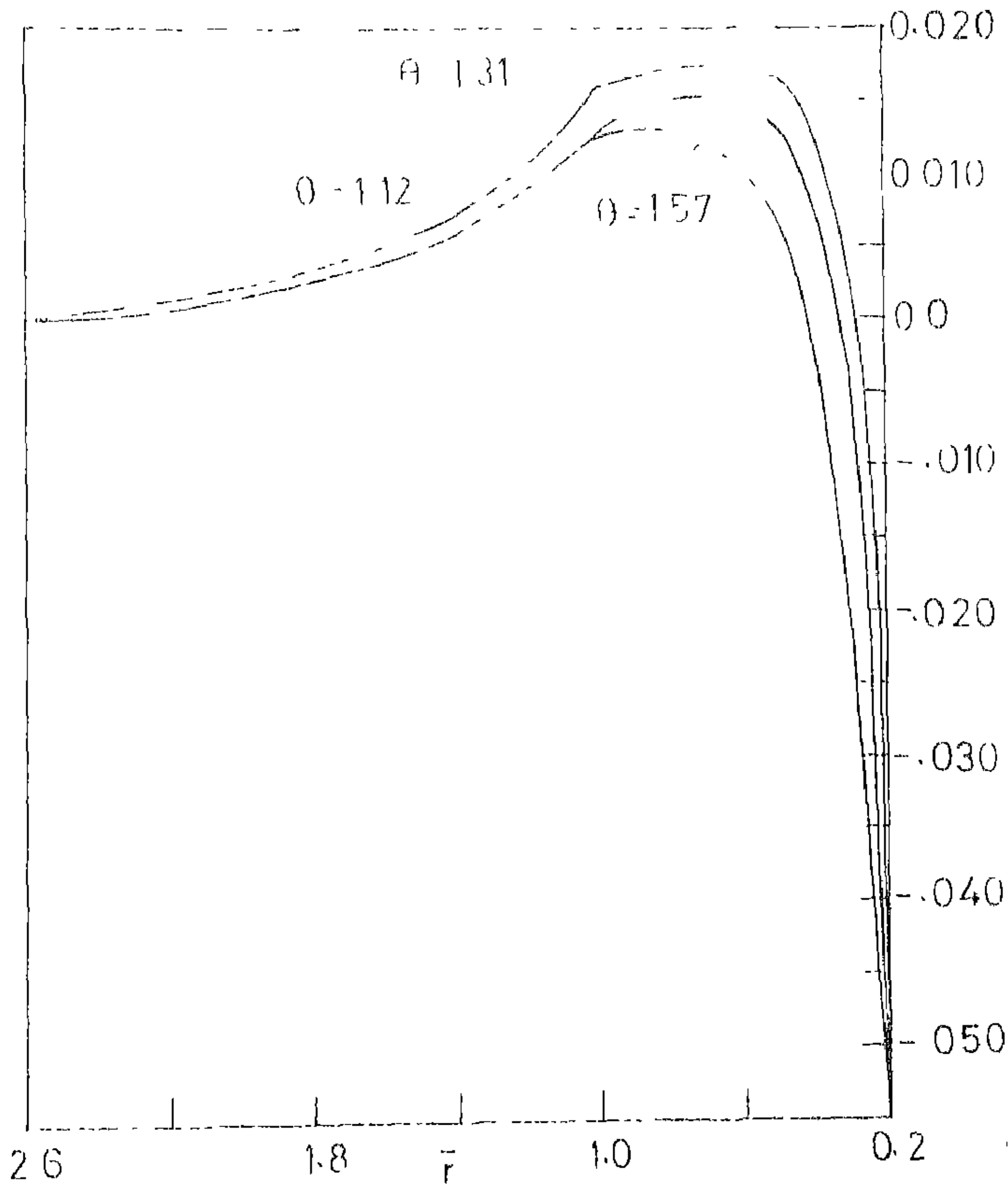

FIG. 4.17 FREE CONICAL SHELL, MOMENT VARIATION OF $\bar{M}_1$ FOR DIFFERENT θ VALUES

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

5.1 CONCLUSIONS:

Foundation model in orthogonal curvilinear coordinate system, in which the displacements in all the three orthogonal directions have been included, has been developed using variational method. A wide variety of problems of elastic foundations can be solved using this model because of its general nature.

Problems of shallow spherical and conical shell on elastic foundations have been solved. All the three important types of boundary conditions, namely simply supported, fixed and free, have been considered. In case of free boundary conditions flank (foundation beyond shell) have been considered and an effective flank width which needs to be considered has been suggested.

Kalnins multisegment technique which has been used to solve the boundary value problems, has worked very well and results obtained during forward integration have matched upto seven significant figure the results obtained on backward integration.

The results obtained for the numerical examples have been discussed at the end of the Chapter III and IV. Numerical results for all the variables and for all the cases considered have not been presented, however important results have been presented to give both quantitative and qualitative presentation of forces and displacements for all the cases.

Although no cost analysis has been done but it can be easily seen from Figs. (3.7), (3.8) and (3.9) that shells would be much more efficient as foundation of column than the plates are.

One important observation that can be made from Figs. (3.3a), (3.3b), (3.4a) and (3.4b) is that the shell slips over the foundation. This happens because no friction has been assumed between shell and foundation. It can be noted from the variation of q that it is not constant over the depth of foundation.

In the numerical examples, thickness of the foundation has been considered as finite and the foundation has been assumed to be made of single material but foundation of infinite thickness and of different elastic layers can be considered with equal ease by proper choice of functions of normal distribution of displacements.

5.2 RECOMMENDATIONS FOR FURTHER WORK.

In the author's view following are the areas where research work are required.

(a) Thick shells on elastic foundations.

(b) Dynamic analysis of shells on elastic foundation,

(c) Use of the proposed model on other types of shells like hypar.

(d) To consider certain amount of friction between shell and foundation to make the analysis more realistic.

In nut shell, good amount of research work, both analytical and experimental is required on shells on elastic foundations before design charts can be prepared.

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